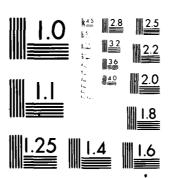
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COMPUTER MODELS FOR CONDUCTING SURFACES

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RADC-TR-81-222 Final Technical Report August 1981



COMPUTER MODELS FOR CONDUCTING SURFACES

Worcester Polytechnic Institute

Jitendra Singh

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Under this effort work has been done to derive and use closed form expressions for generalized impedance parameters involving rectangular, conducting patches in free space. A Galerkin model for the interaction between two wires, a wire and a patch and two patches has been developed. Basis functions sinusoidal in the direction of the current and triangular in the transverse direction have been used. Farrar's Integration technique has been used so as to give most of the results of

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ABSTRACT

Under this effort work has been done to derive and use closed form expressions for generalized impedance parameters involving rectangular, conducting patches in free space.*

A Galerkin model for the interaction between two wires, a wire and a patch and two patches has been developed. Basis functions sinusoidal in the direction of the current and triangular in the transverse direction have been used. Farrar's Integration technique has been used so as to give most of the results of the model in closed form.

The original proposal called for the patches to be parallel. This restriction has not been applied and the formulation is general with respect to the relative orientations of the transmitter subsection and the receiver subsection.

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CHAPTER 1: FORMULATION OF THE PROBLEM

1.0 Introduction

The treatment of conducting surfaces using the method of moments has been a complex problem. Wire grid models have been used with some success and are currently available in the GEMACS system (Balestri et. al. (1977)), among others. Wire gridding gives good results for far field problems and scattering problems. If impedance, currents or near fields are required, or if the surface contains slots comparable to the size of the grid, the results from wire gridding are poor. Wire grid models have been criticized because of questions of their validity (see, for example, Lee et. al, 1976). Surface current models have been developed by Knepp and Goldhirsh (1972)., Albertson et. al. (1974), Wang (1974), Wilton et. al. (1976), and Singh (1977a). Both Knepp and Albertson et. al. use the magnetic field formulation with pulse basis functions and point matching. Wilton, et. al. use the electric field integral equation with pulse basis functions and point matching.

Wang uses the electric field formulation. His expansion functions are sinusoidal in the direction of the current and uniform in the transverse direction. The same applies to his testing functions. The details of his impedance calculation are not given.

Singh has treated surface patches where both basis and testing functions are sinusoidal in the direction of the current as well as

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in the transverse direction. In addition, non-rectangular shapes of basis function patches are permitted.

The limiting factor in all techniques mentioned above is the large amount of CPU time required to numerically perform up to four integrations for each matrix element and to solve the resulting matrix equations. It is to be noted that if ℓ wavelengths is some average linear dimension of a surface and if ν basis functions per wavelength are to be used, the core required for the matrix is of the order of $(\ell\nu)^4$. The CPU time required to fill such a matrix is also of the order of $(\ell\nu)^4$ and the CPU time required to solve the equations is of the order of $(\ell\nu)^6$. Such high order dependence may result in CPU times of the order of several hours for problems of moderate size (about 2 or 3 wavelengths square).

Recently, a new integration technique due to Farrar (1978) et. al. has been published. This technique has been used to perform some of the integrations in closed form and thereby reduce computation times drastically.

The problem has been formulated in the next three subsections. Subsection 1.4 is devoted to defining the notation for the coord-inate system to be used in the model which is developed in Chapter 2.

1.1 Basis Functions for Wires

Consider a wire subsection i extending from $\ell = -\ell_1$ to $\ell = +\ell_2$. The (current) basis function $\phi_1(\ell)$ is given in terms of ℓ_1 and ℓ_2 as

$$\phi_{\mathbf{i}}(\ell) = \begin{cases} \frac{\sin k(\ell + \ell_1)}{\sin k\ell_1} & -\ell_1 \le \ell \le 0 \\ \frac{\sin k(\ell_2 - \ell)}{\sin k\ell_2} & 0 < \ell \le \ell_2 \end{cases}$$

$$0 \quad \text{elsewhere}$$
(1.1.1)

where

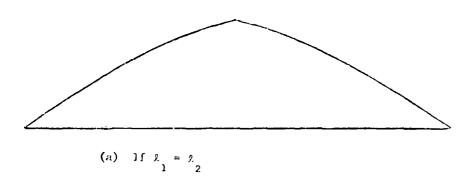
$$k = \frac{2\pi}{\lambda}$$
 is the wave number.

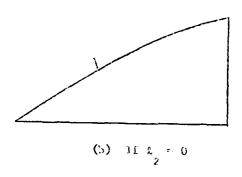
Lengths ℓ_1 , ℓ_2 are either equal or one of them is zero. Basis functions $\phi_i(\ell)$ are pictorially depicted in figure 1.1.1.

Figure 1.1.2 shows a wire divided into several basis functions. The presence of a half sinusoid at the end, if that end forms a junction, may be noted. The half sinusoid implies the presence of a point charge at the end. The expressions in the next sections take this point charge into account. $\phi(\ell)$ is dimensionless. The current on the wire segments in the configuration is given by

$$\sum_{i} \phi_{i}(\ell) I_{i} \underline{u}_{\ell_{i}}$$

where I is the current on wire segment i, \underline{u}_{i} is the unit vector in the ℓ direction for wire segment i.





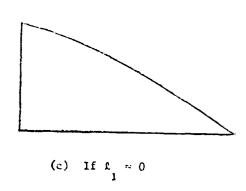
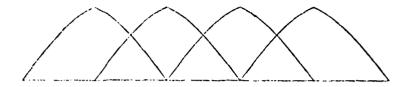
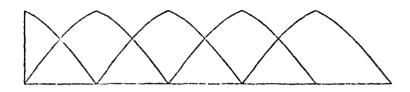


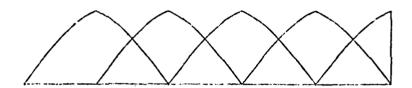
Figure 1.1.1 Basis functions for wires



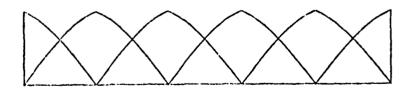
(a) If both ends are open



(b) If the start end is at a junction



(c) If the finish end is at a junction



(d) If both ends are at junctions

Figure 1.1.2 Division of a wire into basis functions.

1.2 Basis Functions for Surfaces

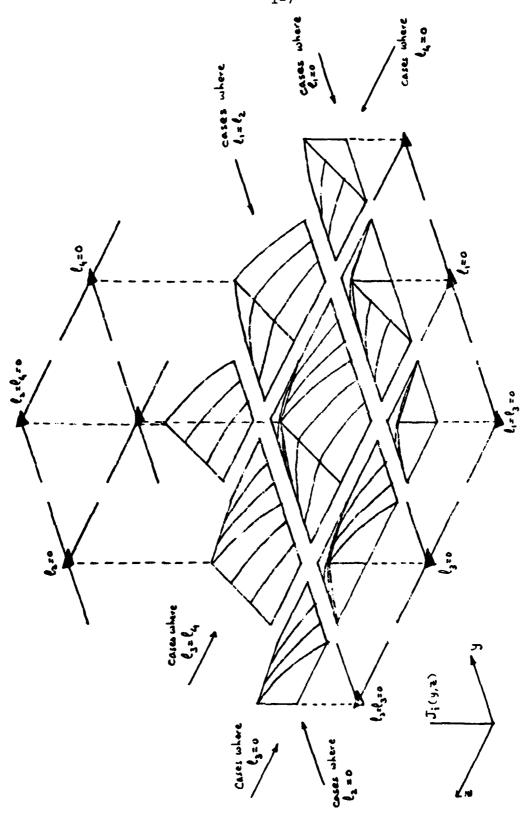
Consider a surface subsection i carrying current in the u direction and extending $y = -\ell_3$ to ℓ_4 and $z = -\ell_1$ to ℓ_2 . The current density basis function for $\underline{J}_i(y,z)$ is given in terms of ℓ_1 , ℓ_2 , ℓ_3 and ℓ_4 as $\psi_i(y)\phi_i(z)\underline{u}_z$ where

$$\psi_{\mathbf{i}}(\mathbf{y}) = \begin{cases} \frac{\ell_3 + \mathbf{y}}{\ell_3} - \ell_3 \le \mathbf{y} \le 0 \\ \frac{\ell_4 - \mathbf{y}}{\ell_4} & 0 < \mathbf{y} \le \ell_4 \\ 0 & \text{elsewhere} \end{cases}$$
 (1.2.1)

Lengths ℓ_3 and ℓ_4 are either equal or one of them is zero. It is to be noted that the (x,y,z) coordinate system is local to the surface subsection under consideration and is defined in such a way as to have the current in the subsection flowing in the z direction. An overlapping subsection carrying current in the orthogonal direction is a part of the set of basis functions. Thus the method of moments solution may be expected to contain current density components in any direction.

Depending upon the values of ℓ_1 , ℓ_2 , ℓ_3 and ℓ_4 , nine different types of basis functions are possible. These and their two dimensional representations are shown in figure 1.2.1. Figure 1.2.2 shows the complete set of basis functions for a surface ABCD whose edges AC and CD are connected to other planes. The current density on the surface segments in the configuration is given by

^{*}In this report, the term "surface patch" implies a surface expansion function and not a physical section of the surface. It is assumed that a local coordate system has been arranged so as to have the patch in the yz plane, the z direction being the direction of the current.



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Figure 1.2.1: Pictorial (3D) representation of all possible current density basis function types possible. Some of the corresponding 2D representations are drawn below the 3D representation and some others are drawn above. The 2D representations and the corresponding 3D representations are connected by dotted lines.

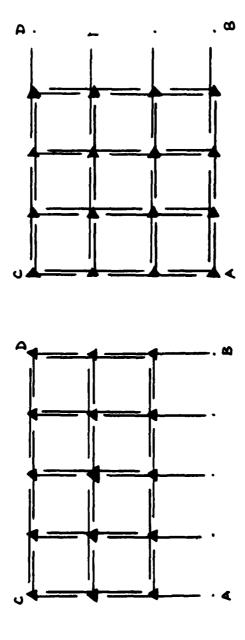


FIGURE 1.2.2: A full set of basis functions for a surface ABCD whose edges AC and CD are connected to other planes. The surface is drawn twice to show the orthogonal current densities separately.

$$\sum_{i} J_{i} = \sum_{i} \psi_{i}(y) \phi_{i}(z) J_{i} \underline{u}_{i}$$
 (1.2.2)

where J_i is the contribution of the current density basis function $\psi_i(y)\phi_i(z)\underline{u}_{z_i}$ to the total current distribution. All J_i are unknowns in the problem and have units of amps m⁻¹. ψ_i and ϕ_i are dimensionless functions.

1.3 Method of Moments Formulation

The geometry of the problem under consideration may involve wires and surfaces. The unknown in the case of wires is current whereas the unknown in the case of surfaces is current density. The method of moments yields the following system of linear equations:

$$\begin{bmatrix} \begin{bmatrix} z^{WW} \end{bmatrix} & \begin{bmatrix} z^{WS} \end{bmatrix} \\ \begin{bmatrix} z^{SW} \end{bmatrix} & \begin{bmatrix} z^{SS} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \\ \begin{bmatrix} J \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} v^{W} \end{bmatrix} \\ \begin{bmatrix} v^{S} \end{bmatrix} \end{bmatrix} . \tag{1.3.1}$$

The superscripts W or S indicate whether the submatrices in question pertain to wires or surfaces. Furthermore,

$$V_{i}^{W} = \int \underline{E}(\ell) \cdot \underline{u}_{\ell_{i}} \phi_{i}(\ell) d\ell \qquad (1.3.2)$$

and

$$V_{i}^{S} = \iint \underline{E}(y,z) \cdot \underline{u}_{z_{i}} \phi_{i}(z) \psi_{i}(y) dz dy \qquad (1.3.3)$$

where $\underline{E}(.)$ is the incident electric field. It is to be noted that V_{i}^{S} has units of volt-m. It may be noted also that whereas Z^{WW} has units of ohms, Z^{WS} and Z^{SW} have units of ohm-m and Z^{SS} has units of ohm-m². The mutual impedance Z_{ij}^{WW} between two wire segments (at y'=0 and at $\eta=0$) shown in figure 1.3.1 is given by

$$Z_{ij}^{WW} = \int_{-\ell_1}^{\ell_2} \phi_i(\zeta) \underline{u}_{\zeta} \cdot \int_{-\ell_1'}^{\ell_2'} \overline{G}(0,\zeta;0,z') \cdot \phi_j(z') \underline{u}_{z}, dz'd\zeta .$$
(1.3.4)

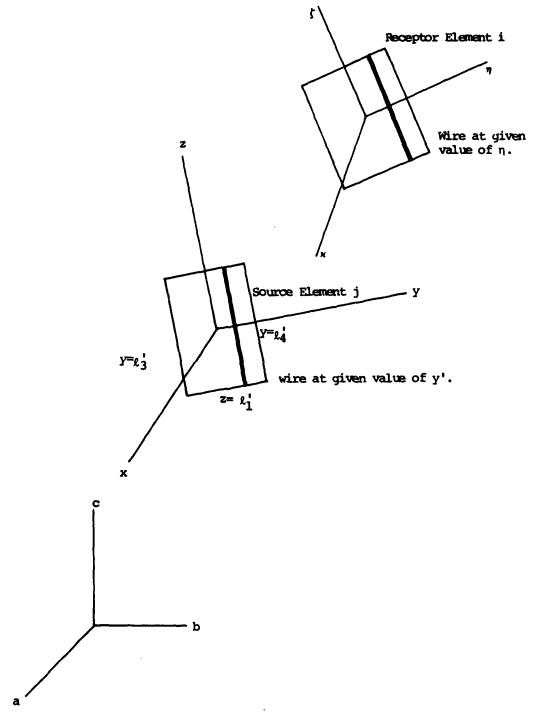


Figure 1.3.1: The physical arrangement for computing mutual impedance. Note that x,y,z and x,η,ζ are local coordinate systems. The global coordinate systems is a,b,c.

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Here, $\overline{G}(0,\zeta;0,z')$ is the free space dyadic Green's function and has units of ohm-m⁻². The mutual impedance Z_{ij}^{WS} between a wire and a surface is given by

$$z_{ij}^{WS} = \int_{-\ell_1}^{\ell_2} \phi_i(\zeta) \underline{u}_{\zeta}. \int_{-\ell_3}^{\ell_4} \psi_j(y') \int_{-\ell_3}^{\ell_2} \overline{G}(0,\zeta;y',z').\phi_j(z')\underline{u}_{z'} dz'dy'd\zeta$$

$$(1.3.5)$$

Similarly,

$$z_{ij}^{SW} = \int_{\ell_3}^{\ell_4} \psi_i(\eta) \int_{-\ell_1}^{\ell_2} \phi_i(\zeta) \underline{u}_{\zeta} \cdot \int_{\ell_1^*}^{\ell_2^*} \overline{G}(\eta,\zeta;0,z') \cdot \phi_j(z') \underline{u}_{z'} dz' d\zeta d\eta$$
and finally,
$$z_{ij}^{SS} = \int_{-\ell_3}^{\ell_4} \psi_i(\eta) \int_{-\ell_1^*}^{\ell_2} \phi_i(\zeta) \underline{u}_{\zeta} \cdot \int_{-\ell_3^*}^{\ell_4^*} \psi_j(\gamma') \int_{-\ell_1^*}^{\ell_2^*} \overline{G}(\eta,\zeta;\gamma',z')$$

$$\phi_j(z') \underline{u}_{z'} dz' d\gamma' d\zeta d\eta$$

$$(1.3.6)$$

In the above equations, the element i has been used as the receptor element and j has been assumed to be the source element. Because of the symmetry of the equations, this choice is arbitrary. In the next chapter, techniques for evaluating the impedance quantities are studied.

1.4 A Notation for the Coordinate Systems

Figure 1.3.1 shows a receiver and a transmitter for which the impedance model is to be developed.

In general, the coordinates and distances from a point (0,y',z') on the (x,y,z) system to a point $(0,n,\zeta)$ on the (χ,η,ζ) system are of interest. Of interest also are some of these values when one or more elements of the set $\langle y',\eta,\zeta \rangle$ are zero. A three subscript notation is used, each subscript being a 0 or a 1. If a subscript is 0, the corresponding variable in the set $\langle y',\eta,\zeta \rangle$ is 0. If the subscript is 1, the corresponding variable in the set is present. Thus, r_{000} is the distance between (0,0,z') on the (x,y,z) system and (0,0,0) on the (χ,η,ζ) system. Similarly, r_{011} is the distance between (0,0,z') on the (x,y,z) system and $(0,\eta,\zeta)$ on the (χ,η,ζ) system. This notation is extended to allow the value (-1) for the subscripts. Under this extension, $r_{01}(-1)$ is the distance between (0,0,z') on the (x,y,z) system and $(0,\eta,-\zeta)$ on the (χ,η,ζ) system. Thus, when the subscript is -1, the corresponding variable in the set $\langle y',\eta,\zeta \rangle$ is negated.

The distance from (0,y',z') to $(0,\eta,\zeta)$ is given by the vector

$$\underline{r}_{111} = (x_{111}, y_{111}, z_{111}) = \underline{r}_{000} + \underline{u}_{y}, y' + \underline{u}_{\eta}^{\eta} + \underline{u}_{\zeta}^{\zeta}$$
 where

$$\underline{u}_{y} = (0, -1, 0),$$

$$\underline{u}_{\eta} = (a_{x\eta}, a_{y\eta}, a_{z\eta})$$
and $\underline{u}_{\zeta} = (a_{x\zeta}, a_{y\zeta}, a_{z\zeta})$ (1.4.2)

In the above equation, \underline{r}_{000} is the vector from the point (0,0,z') on the (x,y,z) system to the origin of the (χ,η,ζ) coordinate system. $a_{\chi\zeta}$, $a_{\gamma\zeta}$, $a_{\zeta\zeta}$, $a_{\chi\eta}$, $a_{\gamma\eta}$, $a_{\zeta\eta}$ are the directional cosines between the respective coordinate axes. Following the properties of directional cosines,

$$a_{x\zeta}^2 + a_{y\zeta}^2 + a_{z\zeta}^2 = 1,$$
 (1.4.3a)

$$a_{x\eta}^2 + a_{y\eta}^2 + a_{z\eta}^2 = 1.$$
 (1.4.3b)

Quite frequently, it is desired that some of the distances be expressed as a quadratic trinomial in one of the variables. A notation for systematically doing so is presented here. The starting point under this notation is always \mathbf{r}_{111} which may be written as

$$r_{111}^{2} (y', \eta, \zeta) \stackrel{\triangle}{=} R = (\underline{r}_{110} + \underline{u}_{\zeta}\zeta) \cdot (\underline{r}_{110} + \underline{u}_{\zeta}\zeta)$$

$$= (r_{110}^{2} + 2\underline{r}_{110} \cdot \underline{u}_{\zeta}\zeta + \zeta^{2}) \stackrel{\triangle}{=} (R_{0}^{\zeta} + R_{1}^{\zeta} \zeta + \zeta^{2}) \cdot (1.4.4a)$$

The arguments of r... may be omitted if they are the variables y',η and ζ . The ζ in equation 1.4.4 is a superscript, not power. Similarly,

$$r_{111}^2 = (R_0^{Y'} + R_1^{Y'} Y' + Y'^2)$$
 and (1.4.4b)

$$r_{111}^2 = (R_0^{\eta} + R_1^{\eta} \eta + \eta^2)$$
 (1.4.4c)

Each of the coefficients of the quadratic trinomials $R_i^{(\cdot)}$ can themselves be written as polynomials of degree (2-i) in other variables. Thus, for example,

$$\mathbf{r}_{110}^{2} (\mathbf{y}', \eta) \triangleq \mathbf{R}_{0}^{\zeta} = (\underline{\mathbf{r}}_{100} + \underline{\mathbf{u}}_{\eta} \eta) \cdot (\underline{\mathbf{r}}_{100} + \underline{\mathbf{u}}_{\eta} \eta) =$$

$$(\mathbf{r}_{100}^{2} + (2\underline{\mathbf{r}}_{100} \cdot \underline{\mathbf{u}}_{\eta}) \quad \eta + \eta^{2}) \triangleq \mathbf{R}_{00}^{\zeta \eta} + \mathbf{R}_{01}^{\zeta \eta} \eta + \eta^{2}$$

$$(1.4.5)$$

Table 1.4.1 shows all such expansions that have been used in this report. They have been derived using a development similar to 1.4.4a, above.

Whenever the ensuing text calls for a coefficient one of whose subscripts is 2, the coefficient is assumed to have the value 1. Whenever the sum of the subscripts is greater than 2, the value of the coefficient is 0.

The vector $\varrho_{111} = (x_{111}, y_{111}, 0)$ is also of importance in the calculations of impedance in cases where $a_{zz} \neq 1$.

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$$\underline{\rho}_{111} = \underline{\rho}_{000} + \underline{v}_{v'} \quad y' + \underline{v}_{n} \eta + \underline{v}_{c} \zeta \tag{1.4.6}$$

where

$$\underline{v}_{y}$$
, = (0, -1,0), \underline{v}_{η} = ($a_{x\eta}$, $a_{y\eta}$, 0), \underline{v}_{ζ} = ($a_{x\zeta}$, $a_{y\zeta}$, 0). (1.4.7)

Expansions for ρ that are similar to those given above for $r_{\mbox{\footnotesize{111}}}$ are possible and are represented by

$$\rho_{111}^2 = \Lambda = (\Lambda_0^{\zeta} + \Lambda_1^{\zeta} \zeta + \zeta^2) v_{\zeta}^2$$
 (1.4.8)

The expansions are similar to those for r_{111} except that none of the coefficients of the trinomial is known apriori to be 1. It is convenient to express the trinomial using the multiplier v_ζ^2 (as has been done in equation 1.4.8), when expansions about ζ are involved. It is more convenient to express it using three coefficients (as has been done in the expansions for Λ_0^ζ in table 1.4.2) when expansions about η or γ' are involved. All expansions are listed in table 1.4.2. It is noted that most expansions do not exist when $\underline{v}_\zeta=0$.

Finally, the quantities $a_{\rho\zeta}$ and cos θ_{111} are expressed as a function of ζ thus:

$$a_{\rho\zeta} = \frac{\frac{\rho_{111} \cdot \underline{u}_{\zeta}}{|\rho_{111}|} - \frac{\frac{(\rho_{110} + \underline{v}_{\zeta}^{\zeta}) \cdot \underline{u}_{\zeta}}{|\underline{v}_{\zeta}| (\Lambda_{0}^{\zeta} + \Lambda_{1}^{\zeta}\zeta + \zeta^{2})^{1/2}} - \frac{\underline{v}_{\zeta}^{2} (\frac{1}{2} \Lambda_{1}^{\zeta} + \zeta)}{|\underline{v}_{\zeta}| (\Lambda_{0}^{\zeta} + \Lambda_{1}^{\zeta}\zeta + \zeta^{2})^{\frac{1}{2}}} = \frac{|\underline{v}_{\zeta}| (\zeta + \frac{1}{2} \Lambda_{1}^{\zeta})}{(\Lambda_{0}^{\zeta} + \Lambda_{1}^{\zeta}\zeta + \zeta^{2})^{\frac{1}{2}}}$$
(1.4.9)

$$\cos (\theta_{111}) = \frac{z_{111}}{r_{111}} = \frac{z_{110} + a_{z\zeta}}{r_{111}}$$
 (1.4.10)

. .

TABLE 1.4.1: Expansion of the Distance Parameters into Quadratic
Trinomials

Coeffici	ient Expansion	Interpretation
R	$R_0^{\zeta} + R_1^{\zeta} \zeta + \zeta^2$	r ₁₁₁
R_0^{ζ}	$R_{00}^{\zeta Y'} + R_{01}^{\zeta Y'} y' + y'^{2}$	r ₁₁₀
R_0^{ζ}	$R_{00}^{\zeta\eta} + R_{01}^{\zeta\eta} + \eta^2$	r ₁₁₀
\mathtt{R}_1^{ζ}	$R_{10}^{\zeta y'} + R_{11}^{\zeta y'} y'$	2 <u>r</u> ₁₁₀ · <u>u</u> ζ
R_1^{ζ}	$R_{10}^{\zeta\eta} + R_{11}^{\zeta\eta} \eta$	2 <u>r</u> 110 · <u>u</u> 5
R ₀₀	$R_{000}^{\zeta y^* \eta} + R_{001}^{\zeta y^* \eta} \eta + \eta^2$	r ₀₁₀
R ₀₁ '	$R_{010}^{\zeta y^{\bullet}\eta} + R_{011}^{\zeta y^{\bullet}\eta}\eta$	2 <u>r</u> 010 · <u>u</u> y ·
R ₀₀	$R_{000}^{\zeta\eta y'} + R_{001}^{\zeta\eta y'} y' + y'^{2}$	r ₁₀₀
R ₀₁ ^{ζη}	$R_{010}^{\zeta\eta y'} + R_{011}^{\zeta\eta y'} y'$	2 <u>r</u> 100 • <u>u</u> η
R ₁₀ '	$R_{100}^{\zeta y^{\bullet} \eta} + R_{101}^{\zeta y^{\bullet} \eta} \eta$	2 <u>r</u> 010 · <u>u</u> z,
$R_{11}^{\zeta Y'}$	R ^ζ Υ'η 110	2a _y • 7,

TABLE 1.4.1 (continued):

Coeffic	cient Expa	nsion Interpretation
R ^{ζη}	$R_{100}^{\zeta \eta y'} + R_{101}^{\zeta \eta y'} y'$	2 <u>r</u> ₁₀₀ · <u>u</u> _ζ
$R_{11}^{\zeta\eta}$	R ^{ζny'}	2a _{ηζ} (=0)
R ₀₀₀ ^{ζΥ'η}		r ₀₀₀
R ₀₀₁ η		2 <u>r</u> 000 · <u>u</u> n
R ₀₁₀ n		2 <u>r</u> 000 <u>u</u> y'
$R_{011}^{\zeta y'\eta}$		^{2a} ny'
R ^{ζηy'}		r ₀₀₀
R ₀₀₁		² <u>r</u> ₀₀₀ . <u>u</u> _y ,
R ^{ζηγ'}		2 <u>r</u> 000. <u>u</u> n
R ₀₁₁ ^{ζηγ} '		^{2a} ŋy'
R ^{ζy'η}		2 <u>r</u> 000• <u>u</u> ζ

TABLE 1.4.1 (continued):

Coefficient	Expansion	Interpretation
R ^{CY'n} 101		2a _{1}ζ} (= 0)
R ^C Y'n 110		^{2а} у' ^ζ
R ^{cny} 100		2 <u>r</u> 000 · <u>u</u> ζ
R ^{ζηγ'} 101		^{2a} y'ζ
R ^{ζη} γ'		2a _{ηζ} (= 0)

TABLE 1.4.2: Expansion of ρ in Quadratic Trinomials

Coeffic	cient Ex	pansion	Interpretation
Λ	$v_{\zeta}^{2} (\Lambda_{0}^{\zeta} + \Lambda_{1}^{\zeta} \zeta + \zeta^{2})$		ρ <mark>2</mark> 111
۸٥	$\Lambda_{00}^{\zeta Y}' + \Lambda_{01}^{\zeta Y}' Y' + \Lambda_{02}^{\zeta Y}' Y$	2	$\rho_{110/v_{\zeta}}^{2}$
Λ_0^{ζ}	$\Lambda_{00}^{\zeta\eta} + \Lambda_{01}^{\zeta\eta} + \Lambda_{02}^{\zeta\eta}^{2}$		$\rho_{110/v_{\zeta}}^{2}$
Λ_{1}^{ζ}	$\Lambda_{10}^{\zeta y}$ ' + $\Lambda_{11}^{\zeta y}$ ' y'		2 <u>ρ</u> 110·Ψζ/νζ
۸۲	$\Lambda_{10}^{\zeta\eta} + \Lambda_{11}^{\zeta\eta}\eta$		$\frac{2\rho}{110} \frac{v}{\zeta} / v_{\zeta}^2$
Λ <mark>ζΥ'</mark>	$\Lambda_{000}^{\zeta y' \eta} + \Lambda_{001}^{\zeta y' \eta} \eta + \Lambda_{002}^{\zeta y' \eta} \eta$	2	ρ_{010}^2/v_ζ^2
Λ <mark>ζΥ'</mark>	$\Lambda_{010}^{\zeta y^{\bullet} \eta} + \Lambda_{011}^{\zeta y^{\bullet} \eta} \eta$		$\frac{2\rho}{110}$. $\frac{v}{y}$, $\frac{v}{v}$
۸ζΥ'	Λ <mark>ζΥ'</mark> η 020		$v_{\mathbf{y}}^2$,/ v_{ζ}^2
Λ ^{ζη}	$\Lambda_{000}^{\zeta \eta Y'} + \Lambda_{001}^{\zeta \eta Y'} Y' + \Lambda_{00}^{\zeta \eta}$	ny' y' ²	ρ <mark>2</mark> 100/ ^ν ζ
۸ ^۲ η úl	Λ ^{ζηγ'} + Λ ^{ζηγ'} γ'		2 <u>ρ</u> 100· <u>ν</u> η /νζ
Λ <mark>ζη</mark> 02	Λ ^{ζη} Υ' 020		v_{η}^2/v_{ζ}^2

Table 1.4.2 (continued):

Coeffic	cient	Expansion	Interpretation
	$\Lambda_{100}^{\zeta Y^{\dagger}\eta} + \Lambda_{101}^{\zeta Y^{\dagger}\eta}\eta$		$\frac{2\underline{\rho}_{01}}{2\underline{\nu}_{\zeta}}, \underline{\nu}_{\zeta}^{2}/v_{\zeta}^{2}$ $\frac{2\underline{\nu}_{\zeta}}{2}, \underline{\nu}_{\zeta}^{2}/v_{\zeta}^{2}$
Λ ^ζ y' 11	۸۲۲ ۹		
Λ ^{ζη} 10	$\Lambda_{100}^{\zeta\eta\gamma'} + \Lambda_{101}^{\zeta\eta\gamma'} \gamma'$		$^{2}\underline{\rho}_{100} \cdot \underline{v}_{\zeta} / v_{\zeta}^{2}$
Λ ^{ζη} 11	Λ ^{ζη} Υ'		$2\underline{v}_{\eta} \cdot \underline{v}_{\zeta}/v_{\zeta}^{2} (= 0)$
Λ <mark>ζΥ'</mark> η			$^{\rho^{2}_{000}/v^{2}_{\zeta}}$
۸ ^{ζΥ'} η 001			$2\varrho_{000}\cdot \underline{v}_{\eta}/v_{\zeta}^{2}$
Λ ^ζ Υ'η 002			v_{η}^2/v_{ζ}^2
Λ ^ζ Υ' ^η 010			$^{2}\underline{\rho}_{000}\cdot\underline{v}_{y}$, $^{\prime}v_{\zeta}^{2}$
۸ ^ζ Υ' ^η 011			$2\underline{\mathbf{v}}_{\eta}\cdot\underline{\mathbf{v}}_{\gamma}$, $/\mathbf{v}_{\zeta}^{2}$
۸ ^ζ Υ' ^η 020			v_y^2 , $/v_\zeta^2$
Λ ^{ζηΥ} ,			$^{0}_{000}^{2}/v_{\zeta}^{2}$

TABLE 1.4.2 (Continued):

Coefficient	Expansion	Interpretation
Λ ^{ζη} Υ'		$2\underline{c}_{000}\underline{v}_{y}, v_{\zeta}^{2}$
Λ ^{ζη} Υ' 002		v_y^2/v_ζ^2
Λ ^{ζηγ'} 010		$\frac{2}{2} = \frac{v_{1}}{v_{2}} / v_{2}^{2}$
Λ ^{ζη} Υ'		$2\underline{v}_{y}\cdots\underline{v}_{\eta}/v_{\zeta}^{2}$
Λ ^{ζη} Υ΄		v_{η}^2/v_{ζ}^2
Λ ^{ζΥ'η} 100		$2\underline{\varepsilon}_{000} \cdot \underline{v} \neq v_{\xi}^2$
Λ ^ζ Υ' ^η 101		$2\underline{v}_{\eta}\cdot\underline{v}_{\zeta}/v_{\zeta}^{2}$
۸ ^۲ ۲ ^۳ ا 110		$2\underline{v}_{y}$. $\underline{v}_{\zeta}/v_{\zeta}^{2}$
ιζη Υ' 100		$2_{\stackrel{\smile}{=}000} \cdot \underline{v}_{i}/v_{i}^{2}$
ζη γ' 101		$2\underline{v}_{\mathbf{Y}}, \underline{v}_{\zeta}/v_{\zeta}^{2}$
ζηγ' 110		$2\underline{v}_{\eta}\cdot\underline{v}_{\zeta}/v_{\zeta}^{2}$

CHAPTER 2. CLOSED FORM SOLUTIONS FOR IMPEDANCES

2.1 Coupling Between Two Surface Segments

The mutual impedance between two surface segments i and j is given by (see equation 1.3.7)

$$z_{ij}^{SS} = \int_{-\ell_3}^{\ell_4} \psi_i(\eta) d\eta \int_{-\ell_1}^{\ell_2} \phi_i(\zeta) \underline{u}_{\zeta} \cdot d\zeta \int_{-\ell_3}^{\ell_4} \psi_j(y') dy' \int_{-\ell_1}^{\ell_2} \overline{G}(\eta, \zeta; y', z') \cdot \phi_j(z') \underline{u}_{z} \cdot dz'$$

$$(2.1.1)$$

To calculate this mutual impedance we first consider the mutual impedance between the surface segment i and a wire on segment j at a given value of y'. This mutual impedance is given by

$$z_{ij}^{SW}(y') = \int_{-\ell_3}^{\ell_4} \psi_i (\eta) d\eta \int_{-\ell_1}^{\ell_2} \phi_i(\zeta) \underline{u}_{\zeta} d\zeta \int_{-\ell_1}^{\ell_2} \overline{G}(\eta, \zeta; y', z') \cdot \phi_j(z') \underline{u}_{z'} dz'$$
(2.1.2)

From 2.1.1 and 2.1.2 it is easy to see that

$$z_{ij}^{SS} = \int_{-\ell_3^*}^{\ell_4^*} \psi_{j}(y') z_{ij}^{SW}(y') dy'. \qquad (2.1.3)$$

Similarly,

$$z_{ij}^{SS} = \int_{-\ell_3}^{\ell_4} \psi_{i}(\eta) z_{ij}^{WS}(\eta) d\eta$$
 (2.1.4)

where $z_{ij}^{WS}(\eta)$ is the mutual impedance between a wire on surface segment i at a given value of n and the surface segment j. Equations 2.1.3 and 2.1.4 portray the hierarchy between z^{SS} and z^{WS} and z^{SW} in that the integration of the latter with respect to one

of the coordinates gives the former. Thus, finding z^{WS} or z^{SW} is a step towards finding z^{SS} . The computation of z^{WS} and z^{SW} is treated in the next section. The problem of evaluating z^{SS} using equation 2.1.4 is revisited in Section 2.5 following the computation of z^{WS} and z^{SW} in Section 2.4.

2.2 Coupling Between Wire and Surface Segments

Using arguments similar to those used in Section 2.1, it is easily seen that

$$z_{ij}^{SW} = \int_{-\ell_3}^{\ell_4} \psi_{i}(\eta) \quad z_{ij}^{WW}(\eta) d\eta$$
 (2.2.1)

where $Z_{ij}^{WW}(\eta)$ is the mutual impedance between a wire on segment i at η and the wire j. $Z_{ij}^{SW}(\gamma')$ defined in equation 2.1.2 is given by

$$z_{ij}^{SW}(y') = \int_{-k_3}^{k_4} \psi_i(\eta) z_{ij}^{WW}(\eta, y') d\eta$$
 (2.2.2)

where z_{ij}^{WW} (n,y') is the mutual impedance between a wire on segment i at n and a wire on segment j at y'.

Similarly,

$$z_{ij}^{WS} = \int_{-\ell_3}^{\ell_4} \psi_{j}(y') Z_{ij}^{WW}(y') dy', \qquad (2.2.3a)$$

$$z_{ij}^{WS}(\eta) = \int_{-\ell_3}^{\ell_4} \psi_j(y') z_{ij}^{WW}(y', \eta) dy'. \qquad (2.2.3b)$$

Equations 2.2.3 and 2.2.2 portray the hierarchy between z^{WS} and z^{SW} and z^{WW} , in the sense that integration of the latter with respect to y' or n gives z^{WS} or z^{SW} respectively. Thus, finding z^{WW} is a step towards finding z^{WS} or z^{SW} . The computation of z^{WW} is treated in the next section. The problem of evaluating z^{WS} or z^{SW} from z^{WW} is revisited in section 2.4.

2.3 Wire-to-Wire Coupling

Equations 1.3.4 through 1.3.7 contain the term

$$E_{\zeta(i)}/I_{j} = \underline{u}_{\zeta} \cdot \int_{-\ell_{1}^{i}}^{\ell_{2}^{i}} \overline{G}((.), \zeta; (.), z') \cdot \phi_{j}(z') \underline{u}_{z'} dz' = 2.3.1$$

as their innermost integral. This quantity is the f directed component of the electric field at $(0,\eta,\lambda)$ due to a unit current on subsection j and has units of volts m^{-1} amp $^{-1}$. The result of the integration is a vector whose dot product with \underline{u}_{j} gives $E_{\zeta(i)}/I_{j}$. A closed form solution has been derived by Schelkunoff and Friis (1952) and restated in convenient terms by Singh* (1977). When I_{j} is 1, $E_{\zeta(i)}$ is given by

$$\mathbf{E}_{\mathbf{Z}(\mathbf{i})}(\ldots) = -\frac{\partial \mathbf{V}}{\partial \zeta} + \mathbf{a}_{\mathbf{Z}\zeta}\mathbf{E}_{\mathbf{Z}} + \mathbf{a}_{\zeta\zeta}\mathbf{E}_{\zeta}$$
 (2.3.2)

where

$$V(y',\eta,\xi) = -\frac{j}{4\pi\omega} \sum_{\chi'=-\chi'_1,0,\chi'_2} \frac{1}{\chi'_1, |\psi(z')|} \cdot \frac{e^{-jkr_{111}}}{r_{111}},$$
(2.3.3)

$$E_{z}(...) = \frac{j}{4\pi\omega} \sum_{q'=-q'_{1},0,q'_{2}} \left[\frac{\partial \phi(z')}{\partial z'}\right] \frac{e^{-jkr_{111}}}{r_{111}} (2.3.4)$$

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^{*}Pages 40-42.

and

$$E_{\rho}(\ldots) = \frac{k}{4\pi\omega\epsilon\rho} (\ldots) \sum_{\substack{\ell'=-\ell''\\1},0,\ell''\\2} \Delta [\phi(z')] \exp(-jkr_{111})$$

$$+ \frac{j}{4\pi\omega\varepsilon\rho} (...) \sum_{\ell'=\ell_1',0,\ell_2'} \Delta_{\ell'} \left[\frac{\partial\phi(z')}{\partial z'} \right] \cos\theta_{111} \exp(-jkr_{111})$$
(2.3.5)

where

$$\Delta_{\ell'}(.) = (.)_{\ell'+0} - (.)_{\ell'-0}$$
 (2.3.6)

Since Z_{ij}^{WW} is independent of I_{j} , it is computed using $I_{j}=1$ and is given by

$$Z_{ij}^{WW} (y',\eta) = \int_{-\ell_1}^{\ell_2} \phi_i(\zeta) E_{\zeta(i)} (y',\eta,\zeta) d\zeta \qquad (2.3.7)$$

Since the arguments of the functions $z^{WW}(..)$, $\phi(.)$ and $E_{\zeta}(...)$ provide the context for the subscripts i and j, these subscripts are dropped in the ensuing discussion.

Substituting 2.3.3, 2.3.4, and 2.3.5 into 2.3.2 and then substituting 2.3.2 into 2.3.7, the following equation is obtained:

$$Z^{WW}(y',\eta) = \sum_{\ell=-\ell_1',0,\ell_2'} \frac{4}{t} b_t Z_t (\ell',y',\eta)$$
 (2.3.8)

. .

where

$$b_1^{WW} = \frac{j}{4\pi\omega\epsilon} \Delta_{\ell} [\phi(z')]; \qquad (2.3.9a)$$

$$b_{2}^{WW} = \frac{j}{4\pi\omega\epsilon} \Delta_{\ell} \left[\frac{\partial \phi(z')}{\partial z'} \right]; \qquad (2.3.9b)$$

$$b_3^{WW} = \frac{k}{4\pi\omega\epsilon} \Delta_{g'} [\phi(z')]; \qquad (2.3.9c)$$

$$b_{4}^{WW} = \frac{j}{4\pi\omega\epsilon} \Delta_{\ell} \left[\frac{\partial\phi(z^{*})}{\partial z^{*}} \right]$$
 (2.3.9d)

and where

$$z_{1}^{WW} (\ell', y', \eta) = \int_{-\ell_{1}}^{\ell_{2}} \phi(\zeta) \frac{\partial}{\partial \zeta} \left(\frac{\exp(-jkr_{111})}{r_{111}} \right) d\zeta$$
 (2.3.10a)

$$z_2^{WW}(\ell', y', \eta) = \int_{\ell_1}^{\ell_2} a_{z\zeta} \phi(\zeta) \frac{\exp(-jkr_{111})}{r_{111}} d\zeta$$
 (2.3.10b)

$$\mathbf{Z}_{3}^{WW} \left(\ell', y', n\right) = \int_{-\ell_{1}}^{\ell_{2}} \mathbf{a}_{\rho \zeta} \phi(\zeta) \frac{\exp(-jkr_{111})}{\rho_{111}} d\zeta \qquad (2.3.10c)$$

. .

In the above equations, z_1^{WW} is the contribution of the current discontinuity on the source wire. z_2^{WW} is the contribution of the z directed field generated by the source wire. This contribution is zero when the two wires are orthogonal to each other. z_3^{WW} is the contribution of the ρ directed field generated by the source wire segment when it has a current discontinuity at one end. The contribution of z_3^{WW} is zero whenever the two wire segments are parallel or when the source segment has no current discontinuities on it. z_4^{WW} is the contribution of the ρ directed field generated by the source wire. This contribution is zero when the two wires are parallel.

Theorem 2.3.1

Each of the component impedances \mathbf{Z}_1 through \mathbf{Z}_4 can be written as linear combinations of the impedance quantity $\mathbf{Z}_0^{WW}~(\text{l',y',n,v,s,a})~~\text{where}$

$$z_{0}^{WW} (....) = \int_{-\ell_{1}}^{\ell_{2}} \phi(\zeta) \left(\Lambda_{0}^{\zeta} + \Lambda_{1}^{\zeta} \zeta + \zeta^{2} \right)^{\frac{1}{2}\nu} \zeta^{S} r_{111}^{\alpha} = \exp(-ikr_{111}^{\gamma}) d\zeta$$
(2.3.11)

Proof

The proof constitutes finding the linear combinations of $z_0^{WW}(\dots)$ that give the component impedance in question.

$$z_{1}^{WW}(\ldots) = \int_{-\ell_{1}}^{\ell_{2}} \phi(\zeta) \frac{\partial}{\partial \zeta} \left[\frac{\exp(-jkr_{111})}{r_{111}} \right] d\zeta$$

$$= - \int_{-\ell_1}^{\ell_2} \phi(\zeta) (1 + jkr_{111}) (\frac{1}{2} R_1^{\zeta} + \zeta) r_{111}^{-3} \exp(-jkr_{111}) d\zeta$$

$$= -\sum_{g=0}^{1} \sum_{h=0}^{1} (jk)^{g} (jk)^{2} z_{0}^{\zeta} (\dots 0, h, g-3)$$
 (2.3.12a)

$$z_{2}^{WW}(...) = a_{z\zeta} z_{0}^{WW} (...0,0,-1)$$
 (2.3.12b)

$$z_{3}^{WW}(\ldots) = \int_{-\ell_{1}}^{\ell_{2}} \frac{\left(\frac{l_{2}}{k} \Lambda_{1}^{\zeta} + \zeta\right)}{\left(\Lambda_{0}^{\zeta} + \Lambda_{1}^{\zeta} + \zeta^{2}\right)} \phi(\zeta) \exp(-jkr_{111}) d\zeta$$

$$= \sum_{h=0}^{1} \left(\frac{1}{2} \Lambda_{1}^{\chi} \right)^{1-h} Z_{0}^{WW} (\dots (-2), h, 0)$$
 (2.3.12c)

$$Z_{4}^{WW}(...) = \int_{-\ell_{1}}^{\ell_{2}} \frac{\left(\frac{1}{2} \Lambda_{1}^{\zeta} + \zeta\right)}{\left(\Lambda_{0}^{\zeta} + \Lambda_{1}^{\zeta} \zeta + \zeta^{2}\right)} \phi(\zeta) \frac{\left(z_{110} + a_{z_{\zeta}} \zeta\right)}{r_{111}} \exp(-jkr_{111}) d\zeta$$

$$= \frac{1}{2} \sum_{g=0}^{\ell_{2}} \left(z_{110}\right)^{1-g} \left(\frac{1}{2} \Lambda_{1}^{\zeta}\right)^{1-h} a_{z_{\zeta}}^{g} z_{0}^{WW}(...(-2), (g+h), -1)$$

$$(2.3.12d)$$

End of Proof

The use of theorem 2.3.1 suggests figure 2.3.1 as a flowchart of a procedure to calculate z^{WW} . Note that each of the 4 blocks on the right hand side calls for z_0^{WW} .

A closed form solution for \mathbf{Z}_0^{WW} is developed next. In developing this solution, it is assumed that

$$e^{jx} = \sum_{i=0}^{5} a_i (jx)^i$$
 (2.3.13)

where x is less than $\pi/2$. According to Abramowitz and Stegur* this approximation will result in less than .09 percent error. More terms may be included for greater accuracy. However, the errors developed during solving the method of moments equations are likely to undo any potential benefits of greater accuracy.

$$a_0 = 1$$
 $a_1 = 1$ $a_2 = .49670$ $a_3 = .16605$ $a_4 = .03705$ $a_5 = .00761$

^{*}Page 7**6:**

The above integration range is divided into the $(-l_1$ to 0) and (0 to l_2) ranges. In the first, a change of variables from ζ to $-\zeta$ gives

$$Z_0^{WW}(\ldots) = \sum_{\sigma=1}^{2} (-1)^{\sigma S} \int_0^{\ell_{\sigma}} \frac{\sin(k(\ell_{\sigma} - \zeta))}{\sin(k\ell_{\sigma})} \left(\Lambda_0^{\zeta} + (-1)^{\sigma} \Lambda_1^{\zeta} + \zeta^2 \right)^{\frac{1}{2}\nu}$$

$$\times \zeta^{S} \left(r_{11}(-1)^{\sigma} \right)^{\alpha} \exp(-jk \ r_{11}(-1)^{\sigma}) \ d\zeta \qquad (2.3.14)$$

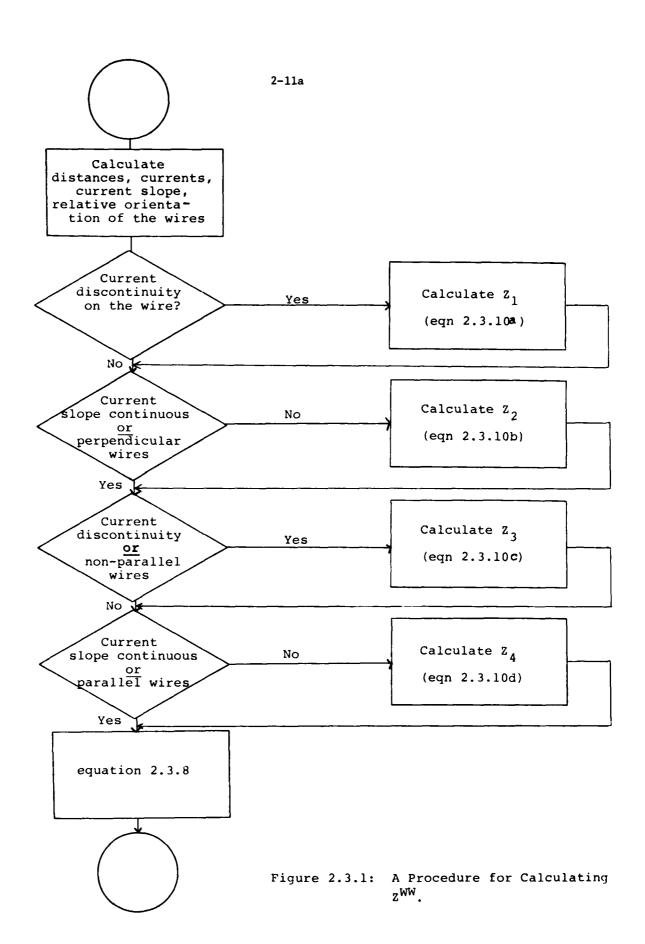
$$= \sum_{\sigma=1}^{2} \frac{(-1)^{\sigma S} \exp(-jkr_{110})}{2j \sin(k\ell_{\sigma})} \int_0^{\ell_{\sigma}} \left(\Lambda_0^{\zeta} + (-1)^{\sigma} \Lambda_1^{\zeta} + \zeta^2 \right)^{\frac{1}{2}\nu} \zeta^{S} \ r_{11}^{\alpha}(-1)^{\sigma}$$

$$x \left[\exp \left[jk \left(\ell_{\sigma} - \zeta - r_{11(-1)}^{\sigma} + r_{110} \right) \right] - \exp \left[jk \left(-\ell_{\sigma} + \zeta - r_{11(-1)}^{\sigma} + r_{110} \right) \right] d\zeta \right]$$

$$(2.3.15)$$

The argument of the exponent is less than $\, \, \iota_{\sigma} \,$ in the domain of integration. Application of equation 2.3.13 gives

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$$Z_{0}^{WW}(...) = \sum_{\sigma=1}^{2} \frac{(-1)^{\sigma s} \exp(-jkr_{110})}{2j \sin k\ell_{\sigma}} \sum_{i=0}^{5} a_{i}(jk)^{i}$$

$$\times \int_{0}^{\ell_{\sigma}} (\Lambda_{0}^{\zeta} + (-1)^{\sigma} \Lambda_{1}^{\zeta} + \zeta + \zeta^{2})^{\frac{1}{2}\nu} \zeta^{s} r_{11}^{\alpha}(-1)^{\sigma}$$

$$\times \left[(\ell_{\sigma} - \zeta - r_{11}(-1)^{\sigma} + r_{110})^{i} - (-\ell_{\sigma} + \zeta - r_{11}(-1)^{\sigma} + r_{110})^{i} \right] d\zeta$$

$$(2.3.16)$$

which by repeated use of the binomial theorem gives

$$z_0^{WW}(\ldots) = \sum_{\sigma=1}^{2} \frac{(-1)^{\sigma s} \exp(-jkr_{110})}{2j \sin k\ell_{\sigma}} \quad \sum_{i=0}^{5} a_i(jk)^{i} \sum_{m=0}^{i} \sum_{n=0}^{m} \sum_{p=0}^{i-m} \binom{i}{m} \binom{m}{n} \binom{i-m}{p}$$

The quantity inside the integral is normalized (i.e., the integration limits are made to be 0 and 1) by writing $\zeta = \ell_\sigma \ x \quad \text{and noting that} \quad r_{11(-1)}{}^\sigma = \left(R_0^\zeta + (-1)^\sigma R_1^\zeta + \zeta^2\right)^{\frac{1}{2}} \text{ to give}$

$$z_0^{WW}(\ldots) = \sum_{\sigma=1}^{2} \frac{(-1)^{\sigma s} \exp(-jkr_{110}) \iota_{\sigma}^{s+1} (\Lambda_0^{\zeta})^{\frac{1}{2}v} r_{110}^{\alpha}}{2j \sin(kl_{\sigma})}$$

$$x \left(\frac{r_{110}}{\ell_{\sigma}}\right)^{n+p} z_{A2\nu(s+m-n)(n+\alpha)} \left([\Lambda], [R] \right)$$
(2.3.18)

where the elements Λ_i of the vector [Λ] and R_i of the vector [R] are given, for $0 \le i \le 2$, by

$$\Lambda_{i} = \frac{1}{\Lambda_{0}} \left((-1)^{\sigma} \ell_{\sigma} \right)^{i} \Lambda_{i}^{\zeta} \quad ; \quad R_{i} = \frac{1}{R_{0}} \left((-1)^{\sigma} \ell_{\sigma} \right)^{i} R_{i}^{\zeta}$$

$$(2.3.19)$$

and where

$$Z_{A2NPQ} ([\Lambda], [R]) = \int_0^1 (\Lambda(x))^{\frac{1}{2}N} x^{P} (R(x))^{\frac{1}{2}Q} dx$$
 (2.3.20)

where

$$\Lambda(x) \stackrel{\Delta}{=} \Lambda_0 + \Lambda_1 x + \Lambda_2 x^2 ; R(x) \stackrel{\Delta}{=} R_0 + R_1 x + R_2 x^2. \text{ When N=0,}$$

 $z_{A2...}$ reduces to the simpler

$$^{2}AlpQ ([R]) = \int_{0}^{1} x^{P} (R(x))^{\frac{1}{2}Q} dx.$$
 (2.3.21)

Techniques for evaluating \mathbf{Z}_{AlPQ} in closed form are detailed in Appendix Al, techniques for evaluating $\mathbf{Z}_{\text{A2NPQ}}$ are detailed in Appendix A2.

A few comments regarding terminology are in order: The unsuperscripted Λ_i and R_i , being elements of the vectors $[\Lambda]$ and [R] are distinct from the superscripted variables defined in Section 1.4 (some of the unsuperscripted Λ_i and R_i are related to the superscripted variables of Section 1.4 by equations 2.3.19). Furthermore, the symbols $[\Lambda]$ and [R], being aggregates of the quantities related with, respectively, the two dimensional distance (ρ) and the three dimensional distance (r) between the receiver and the transmitter, are vectors when the wire-to-wire interaction is under investigation. These symbols represent two dimensional matrices when wire-to-surface and surface-to-wire interactions are considered and three dimensional matrices when surface-to-surface interactions are considered.

In summary, the mutual impedance between two wire segments with sinusoidal basis functions is composed of four impedance quantities (equation 2.3.8) each of which can be written as linear combinations of the impedance quantity \mathbf{Z}_0^{WW} (theorem 2.3.1, equations 2.3.12). The impedance quantity \mathbf{Z}_0^{WW} can, in turn, be written as a linear combination of the quantity $\mathbf{Z}_{\text{A2NPQ}}$ for various values of N, P and Q (equation 2.3.18). Whereas techniques for evaluating $\mathbf{A}_{\text{A2NPO}}$ are examined in Appendix A2, it is noted here

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that z_{A2NPQ} is a dimensionless quantity. Therefore, (from equation 2.3.18), z_0^{WW} has dimensions of (length) $^{S+\nu+\alpha+1}$ which is dimensionally consistent with equations 2.3.8 and 2.3.12. Further details regarding the computation of wire-to-wire impedance are treated in Appendix D.

2.4 Coupling Between a Wire Segment and a Surface Segment

Revisited

From section 2.2,

$$z^{WS}(n) = \int_{-\ell_3}^{\ell_4} \psi(y') z^{WW}(y', n) dy', \qquad (2.4.1a)$$

$$\mathbf{z}^{SW}(\mathbf{y}') = \int_{-\ell_3}^{\ell_4} \psi(\mathbf{n}) \mathbf{z}^{WW}(\mathbf{n}, \mathbf{y}') d\mathbf{n}$$
 (2.4.1b)

Replacing the dummy variable y' or η by -y' or $-\eta$ in the negative part of the integration domain,

$$z^{WS}(n) = \sum_{\tau=3}^{4} \int_{0}^{\ell_{\tau}^{'}} \frac{\ell_{\tau}^{'} - y'}{\ell_{\tau}^{'}} z^{WW} ((-1)^{\tau} y', n) dy' \qquad (2.4.2a)$$

and

$$z^{SW}(y') = \sum_{\xi=3}^{4} \int_{0}^{\ell_{\xi}} \frac{\ell_{\xi} - \eta}{\ell_{\xi}} z^{WW} (y', (-1)^{\xi} \eta) d\eta \qquad (2.4.2b)$$

Using the substitutions $~y'=\ell \, _{\tau}^{+} \, y$ and $\eta=\ell _{\xi} y~$ respectively, the above equations may be rewritten as

$$z^{WS}(\eta) = \sum_{\tau=3}^{4} \sum_{e=0}^{1} (-1)^{e} \ell_{\tau} \int_{0}^{1} y^{e} z^{WW}((-1)^{\tau} \ell_{\tau}^{!} y, \eta) dy \qquad (2.4.3a)$$

and

$$z^{SW}(y') = \sum_{\xi=3}^{4} \sum_{f=0}^{1} (-1)^{f} i_{\xi} \int_{0}^{1} y^{f} z^{WW}(y', (-1)^{\xi} i_{\xi}y) dy$$
 (2.4.3b)

Here the variable y is a dummy variable of integration and is unrelated to the y used in the coordinate systems.

Equation 2.3.8 can be substituted into equations 2.4.3 to give

$$\mathbf{z}^{WS}(\eta) = \sum_{\tau=3}^{4} \sum_{e=0}^{1} (-1)^{e} \mathbf{z}_{\tau}^{*} \qquad \sum_{\ell=-\hat{\mathbf{z}}_{1}^{*},0,\hat{\mathbf{z}}_{2}^{*}} \sum_{t=1}^{4} \mathbf{b}_{t}^{WW} \mathbf{z}_{te}^{WS} \left(\hat{\mathbf{z}}_{\tau}^{*},(-1)^{\tau} \hat{\mathbf{z}}_{\tau}^{*}, \eta\right)$$
(2.4.4a)

and

$$z^{SW}(y^*) = \sum_{\xi=3}^{4} \sum_{f=0}^{1} (-1)^{f} \ell_{\xi} \sum_{\ell'=\ell'_{1},0,\ell'_{2}}^{2} \sum_{t=1}^{4} b_{t}^{WW} z_{tf}^{SW} (\ell',\gamma',(-1)^{\xi} \ell_{\xi})$$
(2.4.4b)

where

$$z_{te}^{WS} (\ell', \ell_{\tau}^{*}, \eta) = \int_{0}^{1} y^{e} z_{t}^{WW} (\ell', \ell_{\tau}^{*} y, \eta) dy$$
 (2.4.5a)

and

$$z_{tf}^{SW} \left(i', y', i_{\xi} \right) = \int_{0}^{1} y^{f} z_{t}^{WW} \left(i', y', i_{\xi} y \right) dy \qquad (2.4.5b)$$

It is to be noted that equations 2.4.5 represent definitions of the quantities Z_{te}^{WS} and Z_{tf}^{SW} and that the factors $(-1)^{\tau}$ and $(-1)^{\xi}$ of equations 2.4.4 are absorbed in the dummy variables ℓ_{τ}^{τ} and ℓ_{ξ} for the purpose of this definition and the analysis that follows.

Theorem 2.4.1a

Each of the component impedances Z_{1e}^{NS} through Z_{4e}^{WS} can be written as linear combinations of the impedance quantity Z_{0e}^{WS} $\{\ell',\ell'_{\tau},\ \eta,\nu,s,\alpha\}$ where

$$z_{0e}^{WS} \left(\ldots\right) = \int_{0}^{1} y^{e} z_{0}^{WW} \left(\ell', \ell_{\tau}^{\prime} y, \eta, \nu, s, \alpha\right) dy \qquad (2.4.6a)$$

Proof

The proof constitutes finding the linear combinations of z_{0e}^{WS} (.....) that give the component impedance in question

$$\begin{split} z_{1e}^{WS}(\dots) &= \int_{0}^{1} y^{e} \ z_{1}^{WW} \ (\ell', \ell_{\tau}^{\prime} y, n) \ dy \\ &= - \int_{0}^{1} y^{e} \sum_{g=0}^{1} \sum_{h=0}^{1} (jk)^{g} \left(\frac{1}{2} R_{10}^{\zeta y'} + \frac{1}{2} R_{11}^{\zeta y'} \ell_{\tau}^{\prime} y\right)^{1-h} \\ &= - \sum_{g=0}^{1} (jk)^{g} \sum_{h=0}^{1} (j^{2})^{1-h} \sum_{h'=0}^{1-h} (R_{10}^{\zeta y'})^{1-h-h'} (R_{11}^{\zeta y'})^{h'} (\ell')^{h'} \\ &= - \sum_{g=0}^{1} (jk)^{g} \sum_{h=0}^{1} (j^{2})^{1-h} \sum_{h'=0}^{1-h} (R_{10}^{\zeta y'})^{1-h-h'} (R_{11}^{\zeta y'})^{h'} (\ell')^{h'} \\ &= x \ z_{0}^{WS}(e+h') \ (\dots \ 0, h, g-3) \end{split}$$

$$z_{2e}^{WS}(...) = a_{z\zeta} z_{0e}^{WS}(...0, 0, -1)$$

$$z_{3e}^{WS}(...) = \frac{1}{\Sigma} (\frac{1}{2})^{1-h} \sum_{h=0}^{1-h} (\Lambda_{10}^{\zeta y})^{1-h-h} (\Lambda_{11}^{\zeta y})^{h} (\Omega_{\tau}^{t})^{h} z_{0}^{WS}(e+h') (...(-2), h, 0)$$

$$z_{4e}^{WS}(...) = \sum_{g=0}^{1} a_{z\zeta}^{g} z_{010}^{1-g} \sum_{h=0}^{1} \sum_{h=0}^{1-h} (\frac{1}{2})^{1-h} (\Lambda_{10}^{\zeta y})^{1-h-h'} (\Lambda_{11}^{\zeta y})^{h'} (\Omega_{\tau}^{t})^{h'} (\Omega_{\tau}^{t})$$

End of Proof

Theorem 2.4.1b

Each of the component impedances z_{1f}^{SW} through z_{4f}^{SW} can be written as linear combinations of the impedance quantity z_{0f}^{SW} (1',y',1,v,s,a) where

$$z_{0f}^{SW}(\ldots) = \int_{0}^{1} y^{f} z_{0}^{WW} (\ell', y' \ell_{\xi} y, v, s, \alpha) dy \qquad (2.4.6b)$$

Proof

The proof constitutes finding the linear combinations of $z_{0\,\mathrm{f}}^{\mathrm{SW}}(\ldots)$ that give the component impedance in question. Similar to equations 2.4.7a,

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$$z_{1f}^{SW}(...) = -\sum_{g=0}^{1} (jk)^{g} \sum_{h=0}^{1} (\frac{1}{2})^{1-h} \sum_{h'=0}^{1-h} (R_{10}^{\zeta\eta})^{1-h-h'} (R_{11}^{\zeta\eta})^{h'} (\ell_{\xi})^{h'}$$

$$z_{0(f+h')}^{SW}$$
 (... 0, h, g-3) (2.4.7ba)

$$z_{2f}^{SW}(...) = a_{Z\zeta} z_{0f}^{SW} (... 0, 0, -1)$$
 (2.4.7bb)

$$z_{3f}^{SW}(...) = \sum_{h=0}^{1} (\frac{1}{2})^{1-h} \sum_{h'=0}^{1-h} (\Lambda_{10}^{\zeta\eta})^{1-h-h'} (\Lambda_{11}^{\zeta\eta})^{h'} (\ell_{\xi})^{2} z_{0(f+h')}^{SW}(...(-2),h,0)$$
(2.4.7bc)

$$z_{4f}^{SW}(...) = \sum_{g=0}^{1} a_{z\zeta}^{g} \sum_{g'=0}^{1-g} (z_{000}^{1-g-g'} (a_{z\eta})^{g'} (\ell_{\xi})^{g'} \sum_{h=0}^{1} (\frac{1}{2})^{1-h}$$

$$\frac{1-h}{\Sigma} (\Lambda_{10}^{\zeta\eta})^{1-h-h'} (\Lambda_{11}^{\zeta\eta})^{h'} (\Omega_{\xi})^{h'} \qquad z_{0 (f+g'+h')}^{SW} (\dots (-2), g+h, -1)$$
(2.4.7_{bd})

Equation 2.4.7bd is dissimilar from 2.4.7ad because z_{110} is a function of η but not of y'. This is easily seen by writing, from equations 1.4.1 and 1.4.2, that

$$z_{110} = z_{010} = z_{000} + a_{z\eta}\eta$$
.

End of Proof

Closed form solutions for 2_{0e}^{WS} are developed by substituting equation 2.3.18 into equation 2.4.6a.

$$z_{0e}^{WS} (....) = \int_{0}^{1} y^{e} z_{0}^{WW} (\ell^{*}, \ell^{*}_{\tau} y, \eta, \nu, s, \alpha) dy$$

$$= \frac{2}{\sigma = 1} \frac{(-1)^{\sigma s} \ell_{\sigma}^{s+1}}{2j \sin k \ell_{\sigma}} \sum_{i=0}^{5} a_{i} (jk \ell_{\sigma})^{i} x$$

$$= \frac{i}{\sigma = 1} \frac{m \cdot i - m}{2j \sin k \ell_{\sigma}} \sum_{i=0}^{5} a_{i} (jk \ell_{\sigma})^{i} x$$

$$= \frac{i}{\sigma = 0} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{i}{m} \right) \left(\frac{n}{n} \right) \left(\frac{i - m}{p} \right) (-1)^{m} \left(1 - (-1)^{i + n + p} \right) z_{0e0}^{WS}$$

$$= \frac{1}{\sigma + 1} \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{i}{m} \right) \left(\frac{n}{n} \right) \left(\frac{i - m}{p} \right) (-1)^{m} \left(1 - (-1)^{i + n + p} \right) z_{0e0}^{WS}$$

$$= \frac{1}{\sigma + 1} \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{i}{m} \right) \left(\frac{n}{n} \right) \left(\frac{i - m}{p} \right) (-1)^{m} \left(1 - (-1)^{i + n + p} \right) z_{0e0}^{WS}$$

$$= \frac{1}{\sigma + 1} \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{i}{m} \right) \left(\frac{n}{n} \right) \left(\frac{i - m}{p} \right) (-1)^{m} \left(\frac{1 - (-1)^{i + n + p}}{n + p} \right) z_{0e0}^{WS}$$

$$= \frac{1}{\sigma + 1} \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{i}{m} \right) \left(\frac{i - m}{p} \right) (-1)^{m} \left(\frac{1 - (-1)^{i + n + p}}{n + p} \right) z_{0e0}^{WS}$$

$$= \frac{1}{\sigma + 1} \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{i}{m} \right) \left(\frac{i - m}{n} \right) \left(\frac{1 - (-1)^{i + n + p}}{n + p} \right) z_{0e0}^{WS}$$

$$= \frac{1}{\sigma + 1} \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{i}{m} \right) \left(\frac{i - m}{n} \right) \left(\frac{1 - (-1)^{i + n + p}}{n + p} \right) z_{0e0}^{WS}$$

where

$$z_{0=0}^{WS} = \int_{0}^{1} y^{e} \left(\Lambda_{0}^{\zeta} \right)^{\frac{1}{2}\nu} r_{110}^{\alpha} \exp(-jk r_{110}) \left(\frac{r_{110}}{\ell_{\sigma}} \right)^{n+p} x$$

$$z_{A2\nu (s+m-n) (n+\alpha)} ([\Lambda], [R]) dy \qquad (2.4.9)$$

For the solution of z_{0e0}^{WS} , the argument of the exponential is first reduced to within $\pi/2$ by writing

$$z_{0e0}^{WS} = \exp(-jk \ r_{0l0}) \left[\Lambda_{00}^{\zeta y} \right]^{\frac{1}{2}\nu} r_{0l0}^{\alpha} \left[\frac{r_{0l0}}{\ell_0} \right]^{n+p} x$$

$$\int_{0}^{1} y^{e} \left[\frac{\Lambda_{0}^{\zeta y}}{\Lambda_{00}^{\zeta y}} \right]^{\frac{1}{2}\nu} \left[\frac{r_{1l0}}{r_{0l0}} \right]^{\alpha+n+p} \exp(-jk(r_{1l0}-r_{0l0}))$$

$$\times {}^{2}A2v(s+m-n)(n+\alpha)$$
 ([A], [R]) dy, (2.4.10)

expanding the exponential in the approximate series and then expanding the binomial $\left(r_{010}-r_{110}\right)^{i1}$ to give

$$z_{0e0}^{WS} = \exp(-jkr_{0l0}) \left(\Lambda_{00}^{\zeta Y'}\right)^{\frac{1}{2}\nu} r_{0l0}^{\alpha} \left(\frac{r_{0l0}}{\ell_{\sigma}}\right)^{n+p} x$$

$$\int_{0}^{1} y^{e} \left(\frac{\Lambda_{0}^{\zeta}}{\Lambda_{00}^{\zeta y}} \right)^{\frac{1}{2}v} \left(\frac{r_{110}}{r_{010}} \right)^{\alpha+n+p+m_{1}} Z_{A2v(s+m-n)(n+\alpha)}([\Lambda],[R]) dy$$
(2.4.11)

The factor containing the integral is

$$\int_{0}^{1} y^{e} \left(\frac{\Lambda_{0}^{\zeta}}{\Lambda_{00}^{\zeta Y^{\dagger}}} \right)^{\frac{1}{2}V} \left(\frac{R_{0}^{\zeta}}{R_{00}^{\zeta Y^{\dagger}}} \right)^{\frac{1}{2}(\alpha+n+p+m_{1})} \int_{0}^{1} \left(\frac{\Lambda}{\Lambda_{0}^{\zeta}} \right)^{\frac{1}{2}V} x^{s+m-n} \left(\frac{R}{R_{0}^{\zeta}} \right)^{\frac{1}{2}(n+\alpha)} dx dy$$

$$= \int_{0}^{1} y^{e} \left(\frac{R_{0}^{\zeta}}{R_{00}^{\zeta Y^{t}}} \right)^{\frac{1}{2}(p+m_{1})} \int_{0}^{1} \left(\frac{\Lambda}{\Lambda_{00}^{\zeta Y^{t}}} \right)^{\frac{1}{2}V} \left(\frac{1}{V_{\zeta}^{2}} \right)^{\frac{1}{2}V} x^{s+m-n} \left(\frac{R}{R_{00}^{\zeta Y^{t}}} \right)^{\frac{1}{2}(n+a)} dx dy$$
(2.4.12)

and thus can be written as $Z_{B2 \vee (s+m-n)} (n+\alpha) = (p+m_1)^{([n],[R])}$ where the elements $\Lambda_{i,j}$ of the matrix $[\Lambda]$ are given, for $0 \le i,j \le 2$, by

$$\Lambda_{ij} = \begin{cases} \frac{1}{\Lambda_{00}^{\zeta Y'}} \left((-1)^{\sigma} \ell_{\sigma} \right)^{i} \left(\ell_{\tau}^{i} \right)^{j} \Lambda_{ij}^{\zeta Y'} & i + j \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

and the elements R_{ij} of the matrix [R] are given by

$$R_{ij} = \begin{cases} \frac{1}{R_{00}^{\zeta Y'}} \left((-1)^{\sigma} \ell_{\sigma} \right)^{i} \left(\ell_{\tau}^{i} \right)^{j} R_{ij}^{\zeta Y'} & i + j \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$(2.4.14a)$$

and where

$$z_{B2NPQMS} ([\Lambda], [R]) \stackrel{\Delta}{=} \int_{0}^{1} y^{M} \begin{pmatrix} 2 \\ \Sigma \\ j=0 \end{pmatrix} y^{j}^{\frac{1}{2}S}$$

$$\mathbf{x} \int_{0}^{1} \left(\sum_{i=0}^{2} \sum_{j=0}^{2} \mathbf{x}^{i} \Lambda_{ij} \mathbf{y}^{j} \right)^{i_{2}N} \mathbf{x}^{p} \left(\sum_{i=0}^{2} \sum_{j=0}^{2} \mathbf{x}^{i} R_{ij} \mathbf{y}^{j} \right)^{i_{2}Q} dx dy$$

$$(2.4.15)$$

For N = 0, $Z_{B2NPQMS}$ simplifies to

$$Z_{\text{B1PQMS}} ([R]) = \int_{0}^{1} y^{\text{M}} \begin{pmatrix} 2 \\ \Sigma \\ j=0 \end{pmatrix} R_{0j} y^{j} \Big|_{1}^{1/2} S \int_{0}^{1} x^{\text{P}} \left(\sum_{i=0}^{2} \sum_{j=0}^{2} x^{i} R_{ij} y^{j} \right)^{1/2} Q$$

$$dx dy$$

$$(2.4.16)$$

The techniques for the calculation of $\rm Z_{B1PQMS}$ and $\rm Z_{B2NPQMS}$ are detailed in Appendix B.1 and B.2 respectively.

Combining 2.4.8 and 2.4.11,

$$z_{0e}^{\text{WS}} = \frac{1}{2j} \exp(-jkr_{010}) \left[\Lambda_{00}^{\zeta y} \right]^{\frac{1}{2}\nu} \qquad \qquad r_{010}^{\alpha} \sum_{\sigma=1}^{2} \frac{(-1)^{\sigma s}}{\sin k \ell_{\sigma}} \ell_{\sigma}^{s+1}$$

$$\left\{ \frac{\mathbf{r}_{010}}{\hat{\ell}_{\sigma}} \right\} \overset{\mathbf{n+p}}{\underset{\mathbf{i}_{1}=0}{\overset{5}{\underset{\mathbf{n}_{1}}{\overset{}}}}} \overset{\mathbf{a}_{1}}{\underset{\mathbf{i}_{1}=0}{\overset{\mathbf{i}_{1}}{\overset{\mathbf{i}_{1}}{\overset{}}}}} \overset{\mathbf{i}_{1}}{\underset{\mathbf{r}_{1}=0}{\overset{\mathbf{r}_{010}}{\overset{\mathbf{i}_{1}}{\overset{}}}}} \overset{\mathbf{i}_{1}}{\underset{\mathbf{n}_{1}=0}{\overset{\mathbf{i}_{1}}{\overset{}}}} \overset{\mathbf{i}_{1}}{\underset{\mathbf{n}_{1}=0}{\overset{\mathbf{i}_{1}}{\overset{}}}} \overset{\mathbf{i}_{1}}{\underset{\mathbf{n}_{1}=0}{\overset{\mathbf{i}_{1}}{\overset{}}}} \overset{\mathbf{i}_{1}}{\underset{\mathbf{n}_{1}=0}{\overset{\mathbf{i}_{1}}{\overset{}}}} \overset{\mathbf{i}_{1}}{\underset{\mathbf{n}_{1}=0}{\overset{\mathbf{i}_{1}}{\overset{}}}} \overset{\mathbf{i}_{1}}{\underset{\mathbf{n}_{1}=0}{\overset{\mathbf{i}_{1}}{\overset{}}}} \overset{\mathbf{i}_{1}}{\underset{\mathbf{n}_{1}=0}{\overset{\mathbf{i}_{1}}{\overset{}}}} \overset{\mathbf{i}_{1}}{\underset{\mathbf{n}_{1}=0}{\overset{\mathbf{i}_{1}}{\overset{}}}} \overset{\mathbf{i}_{1}}{\underset{\mathbf{n}_{1}=0}{\overset{\mathbf{i}_{1}}{\overset{\mathbf{i}_{1}}{\overset{}}}} \overset{\mathbf{i}_{1}}{\underset{\mathbf{n}_{1}=0}{\overset{\mathbf{i}_{1}}{\overset{\mathbf{i}_{1}}{\overset{\mathbf{i}_{1}}{\overset{}}}}} \overset{\mathbf{i}_{1}}{\underset{\mathbf{n}_{1}=0}{\overset{\mathbf{i}_{1}}}{\overset{\mathbf{i}_{1}}{\overset{\mathbf{i}_{1}}{\overset{\mathbf{i}_{1}}}$$

$$\times Z_{B2v(s+m-n)(n+\alpha)e(p+m_1)}$$
 ([A],[R]) (2.4.17a)

The factor $(jkr_{010})^{i}l$ has been rewritten as $(jk\ell_{r})^{i}l$ $(\frac{r_{010}}{\ell_{r}})^{i}l$ in the above equation for two reasons: first, to make the expansion in the y' direction appear to be symmetrical with the expansion in the X direction by writing it in the form $\sum_{i} a_{i} (jk\ell_{\sigma})^{i} \sum_{i} a_{i} (jk\ell_{r})^{i}l$; second to anticipate a result to be derived in Appendix D whereby the partial sum $\sum_{m} \sum_{r} \sum_{m} {i \choose m} {m \choose r} {i-m \choose p} (-1)^{m+m}l (1-(-1)^{i+n+p}) Z_{B2}....$ is found to be of the order $(\frac{\ell_{\sigma}}{r_{010}})^{n+p} (\frac{\ell_{r}}{r_{010}})^{i}l$, thus making all the sums in the above expression relatively insensitive to geometry where a large variation in $(\frac{\ell_{r}}{r})$ may be found.

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Similarly,

$$\begin{split} z_{0f}^{SW} &= \frac{1}{2j} \exp\left(-jkr_{100}\right) \left[\Lambda_{00}^{\zeta\eta} \right]^{\frac{1}{2}\nu} & r_{100}^{\alpha} \sum_{\sigma=1}^{2} \left(-1\right)^{\sigma S} \frac{1}{\sin k\ell_{\sigma}} \ell_{\sigma}^{S+1} \\ & \frac{5}{2} a_{i} \left(jk\ell_{\sigma}\right)^{i} \sum_{m=0}^{2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{i}{m} \right] \left[\frac{m}{n} \right] \left(\frac{i-m}{p} \right] & \left(-1\right)^{m} \left[1 - \left(-1\right)^{i+n+p} \right] \left[\frac{r_{100}}{\ell_{\sigma}} \right]^{n+p} \\ & i = 0 \end{split}$$

$$\sum_{\substack{i_1=0\\i_1=0}}^{5} a_{i_1}(jk\ell_{\xi})^{i_1} (\frac{r_{100}}{\ell_{\xi}})^{i_1} \sum_{\substack{j=0\\m_1=0}}^{5} \binom{i_1}{m_1} (-1)^{m_1} Z_{B2\nu(s+m-n)} (n+\alpha)f(p+m_1)^{([\Lambda],[R])}$$
(2.4.17b)

where the elements Λ_{ij} and R_{ij} of the matrices $[\Lambda], [R]$ are given for $0 = i, j \le 2$, by

$$\Lambda_{ij} = \begin{cases} \frac{1}{\Lambda_{00}^{\zeta\eta}} \left((-1)^{\sigma} \ell_{\sigma} \right)^{i} \left(\ell_{\xi} \right)^{j} \Lambda_{ij}^{\zeta\eta} & i + j \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$(2.4.13b)$$

$$R_{ij} = \begin{cases} \frac{1}{R_{00}^{\zeta\eta}} \left((-1)^{\sigma} \ell_{\sigma} \right)^{i} \left(\ell_{\xi} \right)^{j} R_{ij}^{\zeta\eta} & i+j \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$(2.4.13b)$$

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2.5 Coupling Between Two Surface Segments Revisited

From sections 2.1 and 2.4,

$$z^{SS} = \int_{-\ell_{3}}^{\ell_{4}} \psi(\eta) z^{WS} (\eta) d\eta$$

$$= \sum_{\xi=3}^{2} \sum_{f=0}^{2} (-1)^{f} \ell_{\xi} \int_{0}^{1} z^{f} z^{WS} ((-1)^{\xi} \ell_{\xi} z) dz.$$
(2.5.1)

The above equation has been derived similar to equation 2.4.3b. The variable $z=\frac{\eta}{\ell_{\xi}}$ is a dummy variable of integration and is unrelated to the z used in the coordinate system.

Equation 2.4.4a can be substituted into equation 2.5.1 to give

$$z^{SS} = \sum_{\xi=3}^{4} \sum_{f=0}^{1} (-1)^{f} \chi_{\xi} \sum_{\tau=3}^{4} \sum_{e=0}^{1} (-1)^{e} \chi_{\tau}' \sum_{\ell=-\ell_{1},0,\ell_{2}'} \chi_{\ell}' = \chi_{\ell}',0,\ell_{2}'$$

where

$$z_{\text{tef}}^{SS} \left(\ell', \ell_{\tau}', \ell_{\xi}\right) = \int_{0}^{1} z^{\text{WS}} \left(\ell', \ell_{\tau}', \ell_{\xi^{Z}}\right) dz \qquad (2.5.3)$$

.

Theorem 2.5.1

Each of the component impedances z_{1ef}^{SS} through z_{4ef}^{SS} can be written as linear combinations of the impedance quantity z_{0ef}^{SS} (ℓ ', ℓ_{τ} ', ℓ_{ξ} , ν , s, α) where

$$z_{0ef}^{SS}(....) = \int_{0}^{1} z^{f} z_{0e}^{WS} \left(\ell^{r}, \ell^{r}_{\tau}, \ell^{r}_{\xi} z, \nu, s, \alpha \right) dz$$
 (2.5.4)

Proof

The proof constitutes finding the linear combinations of $z_{0\,\mathrm{ef}}^{SS}$ that give the component impedance in question

$$z_{\text{lef}}^{\text{SS}} = -\sum_{g=0}^{1} (jk)^g \sum_{h=0}^{1} (\frac{1}{2})^{1-h} \sum_{h'=0}^{1-h} (R_{110}^{\zeta y'})^{h'} (\ell_{\tau})^{h'} \sum_{h''=0}^{1-h-h'} (R_{100}^{\zeta y'})^{1-h-h'} (R_{101}^{\zeta y'})^{h''}$$

$$(\ell_{\xi})^{h}$$
 $Z_{0 \in +h'}^{SS}$ (2.5.5a)

$$z_{2ef}^{SS} = a_{z\zeta} z_{0ef}^{SS} (...0,0,-1)$$
 (2.5.5b)

$$\mathbf{z}_{3\text{ef}}^{\text{SS}} = \frac{1}{\sum\limits_{h=0}^{\Sigma}} \ (\frac{1}{2})^{1-h} \ \sum\limits_{h'=0}^{1-h} \ (\Lambda_{110}^{\zeta y' \eta})^{h'} \ (\boldsymbol{\ell}_{1}^{\prime}) \ \sum\limits_{h''=0}^{h'} \ (\Lambda_{100}^{\zeta y' \eta})^{1-h-h'-h''} \ (\Lambda_{101}^{\zeta y' \eta})^{h''} \ (\boldsymbol{\ell}_{\xi})^{h''}$$

$$\times Z_{0 \text{ (e+h') (f+h'')}}^{SS} (...(-2),h,0)$$
 (2.5.5c)

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$$\mathbf{z_{4ef}^{SS}} = \mathbf{\hat{z}} \mathbf{a_{g=0}^{g}} \mathbf{\hat{z}_{g'=0}^{1-g}} \mathbf{z_{000}^{1-g-g'}} \mathbf{a_{z_{\eta}}^{g'}} (\ell_{\xi})^{g'}$$

$$z_{0(e+h')(f+g'+h'')}^{SS}$$
 (2.5.5d)

End of Proof

Closed form solution for z_{0ef}^{SS} is developed by substituting equation 2.4.17a into equation 2 5.4 to give

$$z_{0\text{ef}}^{SS}(\ldots) = \sum_{\sigma=1}^{2} \frac{(-1)^{\sigma S} \ell_{\sigma}^{S+1}}{2j \sin k \ell_{\sigma}} \sum_{i=0}^{5} a_{i} (jk \ell_{\sigma})^{i} \sum_{m=0}^{i} \sum_{n=0}^{m} \sum_{p=0}^{i-m} {i \choose m}$$

$${m \choose n} {i-m \choose p} (-1)^m (1-(-1)^{i+n+p}) \sum_{i_1=0}^5 a_{i_1} (jk) \sum_{m_1=0}^{i_1 i_1} {i_1 \choose m_1} (-1)^{m_1}$$

$$z_{0ef0}^{SS}$$
 (2.5.6)

$$z_{0ef0}^{SS} = \int_{0}^{1} z^{f} \left(\Lambda_{00}^{\zeta y'} \right)^{\frac{1}{2}\nu} r_{010}^{\alpha} \exp \left(-jkr_{010} \right) \left(\frac{r_{010}}{t_{\sigma}} \right)^{n+p} \left(r_{010} \right)^{\frac{1}{2}\nu}$$

$$x Z_{B2\nu(s+m-n) (n+\alpha)e(p+m_1)} \{ [\Lambda], [R] \}_{dz}$$
 (2.5.7)

For the solution of $z_{0\,ef\,0}^{SS},$ the argument of the exponential is first reduced to within $\frac{\pi}{2}$ by writing

$$z_{0ef0}^{SS} = \exp(-jkr_{000}) \left(\Lambda_{000}^{\zeta y' \eta} \right)^{\frac{1}{2}\nu} \quad r_{000}^{\alpha+i_1} \quad \left(\frac{r_{000}}{\ell_{\sigma}} \right)^{n+p}$$

$$\times \int_{0}^{1} z^{\frac{\epsilon}{2}} \left(\frac{\Lambda_{00}^{\zeta y' \eta}}{\Lambda_{000}^{\zeta y' \eta}} \right)^{\frac{1}{2}\nu} \left(\frac{r_{010}}{r_{000}} \right)^{\alpha+n+p+i_1} \exp\left[-jk \left(r_{010} - r_{000} \right) \right]$$

$$x Z_{B2v(s+m-n)(n+\alpha)e(p+m_1)} ([\Lambda], [R])_{dz}$$
 (2.5.8)

and expanding the exponential in the approximate series and then expanding the binomial $\left(1-\frac{r_{010}}{r_{000}}\right)^{12}$ to give

$$z_{0ef0}^{SS} = \exp(-jkr_{000}) \left(\Lambda_{000}^{\zeta Y'\eta} \right)^{\frac{1}{2}\nu} r_{000}^{\alpha+i_1} \left(\frac{r_{000}}{\ell_{\sigma}} \right)^{n+p}$$

$$x = \sum_{i_2=0}^{5} a_{i_2} \left(jkr_{000} \right)^{\frac{i_2}{2}} \sum_{m_2=0}^{\frac{i_2}{2}} \left(\frac{i_2}{m_2} \right)^{(-1)} (-1)^{m_2}$$

$$\int_{0}^{1} z^{f} \left(\frac{\Lambda_{000}^{\zeta Y'\eta}}{\Lambda_{000}^{\zeta Y'\eta}} \right)^{\frac{1}{2}\nu} \left(\frac{r_{010}}{r_{000}} \right)^{\alpha+n+p+i_1+m_2} Z_{B2\nu(s+m-n)(n+\alpha)e(p+m_1)} ([\Lambda], [R])_{dz}$$

$$(2.5.9)$$

The factor containing the integral is

$$\int_{0}^{1} z^{f} \left[\frac{\Lambda_{00}^{\zeta y^{\dagger}}}{\Lambda_{000}^{\zeta y^{\dagger} \eta}} \right]^{\frac{1}{2} \nu} \left[\frac{R_{00}^{\zeta y^{\dagger}}}{R_{000}^{\zeta y^{\dagger} \eta}} \right]^{\frac{1}{2} (\alpha + n + p + i_{1} + m_{2})}$$

$$\int_{0}^{1} y^{e} \left(\frac{R_{0}^{\zeta}}{R_{00}^{\zeta Y'}} \right)^{\frac{1}{2}(p+m_{g})} \int_{0}^{1} \left(\frac{\Lambda}{\Lambda_{00}^{\zeta Y}} \right)^{\frac{1}{2}v} x^{s+m-n} \left(\frac{R}{R_{00}^{\zeta Y'}} \right)^{\frac{1}{2}(n+\alpha)} dx dy dz$$

$$= \int_{0}^{1} z^{f} \left(\frac{R_{00}^{\zeta y'}}{R_{000}^{\zeta y'}}\right)^{\frac{1}{2}(i_{1}-m_{1}+m_{2})} \int_{0}^{1} y^{e} \left(\frac{R_{0}^{\zeta}}{R_{000}^{\zeta y'}}\right)^{\frac{1}{2}(p+m_{1})}$$

$$\int_{0}^{1} \left\{ \frac{\Lambda}{\Lambda^{\zeta Y^{\dagger} \eta}} \right\}^{\frac{1}{2} \nu} x^{s+m-n} \left\{ \frac{R}{R_{000}^{\zeta Y^{\dagger} \eta}} \right\}^{\frac{1}{2} (n+\alpha)} dx dy dz$$
 (2.5.10)

and thus can be written as

$$z_{C2v(s+m-n)(n+a)e(p+m_1)f(i_1-m_1+m_2)}$$
 ([A],[R])

where the elements Λ_{ijk} of the three dimensional matrix [Λ] are given by

$$\Lambda_{ijk} = \frac{1}{\Lambda_{000}^{\zeta \dot{y}^{\dagger} \dot{\eta}}} \qquad \left((-1)^{\sigma} \ell_{\sigma} \right)^{i} \left(\ell_{\dot{\tau}}^{i} \right)^{j} \left(\ell_{\xi} \right)^{k} \Lambda_{ijk}^{\zeta \dot{y}^{\dagger} \dot{\eta}} \quad \text{if } i + j + k \leq 2$$

$$= 0, \qquad \text{otherwise} \qquad (2.5.11)$$

the elements of R_{ijk} of the three dimensional matrix [R] are given by

$$R_{ijk} = \frac{1}{R_{000}^{\zeta y' \eta}} \left((-1)^{\sigma} \ell_{\sigma} \right)^{i} \left(\ell_{\tau} \right)^{j} \left(\ell_{\xi} \right)^{k} R_{ijk}^{\zeta y' \eta} \quad \text{if } i+j+k \leq 2$$

$$= 0 \qquad \qquad \text{otherwise} \qquad (2.5.12)$$

and where

$$Z_{\text{C2NPQMSGH}} ([\Lambda], [R]) = \int_{0}^{1} z^{G} \left(\sum_{k=0}^{2} R_{00k} z^{k} \right)^{\frac{H}{2}} \int_{0}^{1} y^{M} \left(\sum_{j=0}^{2} \sum_{k=0}^{2} R_{0jk} y^{j} z^{k} \right)^{\frac{1}{2}S}$$

$$\times \int_{0}^{1} \left(\sum_{i=0}^{2} \sum_{j=0}^{2} \sum_{k=0}^{2} \Lambda_{ijk} x^{i} y^{j} z^{k} \right)^{\frac{1}{2}N} x^{P} \left(\sum_{i=0}^{2} \sum_{j=0}^{2} \sum_{k=0}^{2} R_{ijk} x^{i} y^{j} z^{k} \right)^{\frac{1}{2}N}$$

$$dx dy dz \qquad (2.5.13)$$

Techniques for the computation of $Z_{\text{C2NPQMSGH}}$ are treated in Appendix C.2. Techniques for the computation of $Z_{\text{C1PQMSGH}}([R])$, being a special case of the above for N = 0, are treated in Appendix C.1.

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APPENDIX A

CLOSED FORM SOLUTION OF SOME SINGLE INTEGRALS

A.1 The Solution for ZAIPQ

Closed form solution for the integral

$$z_{AlpQ} ([R]) = \int_{0}^{1} x^{P} (R(x))^{\frac{1}{2}Q} dx$$

where

$$R(x) = R_0 + R_1 x + R_2 x^2$$

is derived here for various integer values of P and Q. In this derivation, it is known that P \geq 0, R₀ = 1 and 4R₂ \geq R₁².

For Q = 0,

$$z_{A1P0} ([R]) = \frac{1}{P+1}$$
 (A.1.1)

$$z_{A1PQ} = z_{A1P(Q-2)} + R_1 z_{A1(P+1)(Q-2)} + R_2 z_{A1(P+2)(Q-2)}$$

A.1.2)

This equation is used as a recurrence relation for the calculation of $\mathbf{Z}_{\mbox{AlPO}}$ for positive values of Q.

For negative values of Q, a recursion relation to reduce the value of P is derived first:

Integrating by parts,

$$z_{A1PQ} = \frac{x^{P+1}}{P+1} \quad R^{\frac{1}{2}Q} \quad \int_{0}^{1} -\frac{Q}{2(P+1)} \quad \int_{0}^{1} x^{P+1} \quad R^{\frac{1}{2}Q-1}(R_{1}+2R_{2}x) dx$$

$$= \frac{x^{P+1}}{P+1} \quad R^{\frac{1}{2}Q} \quad \int_{0}^{1} -\frac{QR_{1}}{2(P+1)} \quad z_{A1(P+1)(Q-2)} -\frac{QR_{2}}{(P+1)} \quad z_{A1(P+2)(Q-2)}$$

Subtracting A.1.3 from A.1.2 and rearranging gives

$$R_{2} \left(1 + \frac{Q}{P+1}\right) Z_{A1(P+2)(Q-2)} = \frac{x^{P+1}}{P+1} R^{\frac{1}{2}Q} \right]_{0}^{1}$$

$$-\left(1 + \frac{Q}{2(P+1)}\right) R_{1} Z_{A1(P+1)(Q-2)} - Z_{A1P(Q-2)}$$
(A.1.4)

Changing all P to P-2, Q to Q+2, for $P+Q+1\neq 0$,

$$\frac{(P-1)}{(P+Q+1)R_2} Z_{A1(P-2)Q}$$
 (A.1.5)

If P = 1, and $Q \neq -2$, this gives

$$z_{A11Q} = \frac{R^{\frac{1}{2}Q+1}}{(Q+2)R_2} \frac{1}{0} - \frac{R_1}{2R_2} z_{A10Q}$$
If P > 1 and Q\neq - (P+1), (A.1.6)

$$Z_{A1PQ} = \frac{1}{(P+Q+1)R_2} \left[(R(1)^{\frac{1}{2}Q+1} - (P+\frac{1}{2}Q)R_1 Z_{A1(P-1)Q} - (P+\frac{1}{2}Q)R_1 Z_{A1(P-1)Q} \right]$$

$$(P-1)$$
 $Z_{A1(P-2)Q}$ (A.1.7)

For P + Q + 1 = 0, equation A.1.3 rewritten as

$$z_{A1PQ} = \frac{1}{R_2} \left(z_{A1(P-2)(Q+2)} - z_{A1(P-2)Q} - R_1 z_{A1(P-1)Q} \right)$$
(A.1.8)

can be used recursively until P = 0 or 1. $Z_{Al0(-1)}$ and $Z_{Al1(-2)}$ are evaluated separately later in this Appendix.

Special Case

If R_2 << 1 the recursive relationships derived above accumulate errors. For that case, equation A.1.7 rewritten as

$$Z_{A1PQ} = \frac{1}{P+1} \left((R(1))^{\frac{1}{2}Q+1} - (P+\frac{1}{2}Q+2) R_1 Z_{A1(P+1)Q} - (P+Q+3)R_2 Z_{A1(P+2)Q} \right)$$
(A.1.9)

along with the approximation that for $R_2 << 1$, large p

$$z_{A1PQ} \simeq \frac{(R(1))^{\frac{1}{2}}(Q)}{P+1}$$
 (A.1.10)

is used for the evaluation of $\mathbf{Z}_{\mbox{AlPQ}}$ for all P and Q. End of Special Case

The base values to be used for recursion relations A.1.5. through A.1.8. are given in tables of integrals (Gradshteyn and Ryzhik, 1965) and are reproduced here.

$$Z_{A10(-3)} = \begin{cases} \frac{4}{\Delta} \left[\left(R_2 + \frac{1}{2} R_1 \right) R^{-\frac{1}{2}} (1) - \frac{1}{2} R_1 \right] & \text{if } \Delta \neq 0 \\ \frac{R_1 + R_2}{R_1 (R_1 + R_2 + 1)} & \text{if } \Delta = 0 \end{cases}$$
(A.1.11)

where

$$\Delta = 4R_2 - R_1^2 \tag{A.1.12}$$

$$z_{A10(-2)} = \begin{cases} \frac{2}{\sqrt{\Delta}} \left[\tan^{-1} \frac{2R_2 + R_1}{\sqrt{\Delta}} - \tan^{-1} \frac{R_1}{\sqrt{\Delta}} \right] & \text{if } \Delta > 0 \\ \frac{R_1}{R_1 + R_2} & \text{if } \Delta = 0 \end{cases}$$
(A.1.13)

$$z_{A11(-2)} = \frac{1}{2R_2} \ln R(1) - \frac{1}{2} \frac{R_1}{R_2} z_{A10(-2)}$$
 (A.1.14)

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$$Z_{A10(-1)} = \begin{cases} \frac{\ell n}{\sqrt{R_2}} \left(\frac{R_2^{\frac{1}{2}} R^{\frac{1}{2}}(1) + R_2 + \frac{1}{2} R_1}{R_2^{\frac{1}{2}} + \frac{1}{2} R_1} \right) & \text{if } \Delta \neq 0 \\ R_2^{-\frac{1}{2}} \ln \left(\frac{R_2 + \frac{1}{2} R_1}{\frac{1}{2} R_1} \right) & \text{if } \Delta = 0 \end{cases}$$
(A.1.15)

A.2 The Solution for z_{A2NPQ}

The integral Z_{A2NPQ} is evaluated here given that $N \ge -2$, $P \ge 0$, $Q \ge -N-3$ $\Lambda_0 = 1$, $R_0 = 1$, $4\Lambda_2 \ge \Lambda_1^2$, $4R_2 \ge R_1^2$

$$Z_{A2NPQ} \left[\begin{bmatrix} \Lambda \end{bmatrix}, \begin{bmatrix} R \end{bmatrix} \right] = \int_{0}^{1} \left[\begin{bmatrix} 1 \\ x \\ x^{2} \end{bmatrix}^{T} \begin{bmatrix} \Lambda_{0} \\ \Lambda_{1} \\ \Lambda_{2} \end{bmatrix} \right]^{\frac{1}{2}N} x^{P} \left[\begin{bmatrix} 1 \\ x \\ x^{2} \end{bmatrix}^{T} \begin{bmatrix} R_{0} \\ R_{1} \\ R_{2} \end{bmatrix} \right]^{\frac{1}{2}Q} dx$$
(A.2.1)

is reduced by using the recursion relations

$$z_{A2NPQ} = z_{A2(N-2)PQ} + {}^{\Lambda}1^{Z}_{A2(N-2)(P+1)Q} + {}^{\Lambda}2^{Z}_{A2(N-2)(P+2)Q}$$
(A.2.2)

and

$$z_{A2NPQ} = z_{A2NP(Q-2)} + R_1 z_{A2N(P+1)(Q-2)} + R_2 z_{A2N(P+2)(Q-2)}$$
(A.2.3)

until N and Q are either zero or negative. If N=0,

$$Z_{A20 PQ} = Z_{A1PQ}$$
 ([R]). (A.2.4)

If Q = 0

$$z_{A2NP0} = z_{A1PN} ([\Lambda]). \qquad (A.2.5)$$

If both N and Q are negative, a recursion relation on P is derived by integrating A.2.1 by parts to give

•

$$\begin{split} Z_{\text{A2NPQ}} &= \frac{x^{P+1}}{P+1} \left(\Lambda \right)^{\frac{1}{2}N} \left(R \right)^{\frac{1}{2}Q} \bigg]_{0}^{1} \\ &- \frac{\Lambda_{0}R_{1}Q + R_{0}\Lambda_{1}N}{2(P+1)} Z_{\text{A2}(N-2) (P+1) (Q-2)} \\ &- \frac{\Lambda_{1}R_{1}(N+Q) + 2\Lambda_{0}R_{2}Q + 2R_{0}\Lambda_{2}N}{2(P+1)} Z_{\text{A2}(N-2) (P+2) (Q-2)} \\ &- \frac{\Lambda_{1}R_{2}N + R_{1}\Lambda_{2}Q + 2\Lambda_{2}R_{1}N + 2R_{2}\Lambda_{1}Q}{2(P+1)} Z_{\text{A2}(N-2) (P+3) (Q-2)} \\ &- \frac{\Lambda_{2}R_{2}(N+Q)}{P+1} Z_{\text{A2}(N-2) (P+4) (Q-2)} \end{split}$$

Equations A.2.2 and A.2.3 have been used in deriving the above equation wherever necessary. Another equation is obtained by applying equation A.2.3 to each of the terms on the right side of equation A.2.2.

$$z_{A2NPQ} = {}^{\Lambda_0} {}^{R_0} z_{A2(N-2)P(Q-2)}$$

$$+ ({}^{\Lambda_1} {}^{R_0} + {}^{R_1} {}^{\Lambda_0}) z_{A2(N-2)(P+1)(Q-2)}$$

$$+ ({}^{\Lambda_2} {}^{R_0} + {}^{\Lambda_1} {}^{R_1} + {}^{\Lambda_0} {}^{R_2}) z_{A2(N-2)(P+2)(Q-2)}$$

$$+ ({}^{\Lambda_2} {}^{R_1} + {}^{\Lambda_1} {}^{R_2}) z_{A2(N-2)(P+3)(Q-2)}$$

$$+ {}^{\Lambda_2} {}^{R_2} z_{A2(N-2)(P+4)(Q-2)}$$

$$(A.2.7)$$

 Z_{A2NPO} is eliminated from equation A.2.6 and A.2.7 to give

$$\frac{N+P+Q+1}{P+1} = \Lambda_{2}R_{2} Z_{A2 (N-2) (P+4) (Q-2)} = \frac{x^{P+1}}{P+1} (\Lambda)^{\frac{1}{2}N} (R)^{\frac{1}{2}Q} \Big|_{0}^{1}$$

$$- \frac{1}{P+1} \Big[\Lambda_{1}R_{2} (\frac{1}{2}N+P+Q+1) + \Lambda_{2}R_{1} (N+P+\frac{1}{2}Q+1) \Big] Z_{A2 (N-2) (P+3) (Q-2)}$$

$$- \frac{1}{P+1} \Big[\Lambda_{0}R_{2} (P+Q+1) + \Lambda_{1}R_{1} (\frac{1}{2}N+P+\frac{1}{2}Q+1) + \Lambda_{2}R_{0} (N+P+1) \Big]$$

$$Z_{A2 (N-2) (P+2) (Q-2)}$$

$$- \frac{1}{P+1} \Big[\Lambda_{0}R_{1} (P+\frac{1}{2}Q+1) + \Lambda_{1}R_{0} (\frac{1}{2}N+P+1) \Big] Z_{A2 (N-2) (P+1) (Q-2)}$$

$$- \Lambda_{0}R_{0} Z_{A2 (N-2) P (Q-2)} (A.2.8)$$

Changing N to (N+2), P to (P-4), Q to (Q+2), the recursive relationship is (N and Q < 0)

$$\begin{split} z_{\text{A2NPQ}} &= \frac{x^{\text{P-3}}}{\text{N+P+Q+1}} \frac{(\Lambda)^{\frac{1}{2}\text{N+1}} (R)^{\frac{1}{2}\text{Q+1}}}{\Lambda_2 R_2} \bigg|_0^1 \\ &= \frac{\left[\frac{\Lambda_1}{\Lambda_2} \left(\frac{1}{2}\text{N+P+Q}\right) + \frac{R_1}{R_2} \left(\text{N+P+$\frac{1}{2}$Q}\right)\right]}{\text{N+P+Q+1}} z_{\text{A2N(P-1)}Q} \end{split}$$

$$-\frac{\left[\frac{\Lambda_{0}}{\Lambda_{2}}\right]^{(P+Q-1)}+\frac{\Lambda_{1}R_{1}}{\Lambda_{2}R_{2}}\left(\frac{1}{2}N+P+\frac{1}{2}Q-1\right)+\frac{R_{0}}{R_{2}}\left(N+P-1\right)}{N+P+Q+1}}{Z_{A2N(P-2)Q}$$

$$-\frac{\left[\frac{{{\Lambda _0}R_1}}{{{\Lambda _2}R_2}}\left({P + \frac{1}{2}Q - 2} \right) \right. \\ \left. + \frac{{{\Lambda _1}R_0}}{{{\Lambda _2}R_2}}\left({\frac{1}{2}N + P - 2} \right) \right)}{{N + P + Q + 1}} \right. \\ \left. z_{A2N \, (P - 3) \, Q} \right.$$

$$-\frac{(P-3) \Lambda_0 R_0}{(N+P+Q+1) \Lambda_2 R_2} Z_{A2N(P-4)Q} \qquad (A.2.9)$$

This recursive relationship is used while $P \ge 3$ and $N+P+Q+1 \ne 0$. $Z_{\mbox{A2NPQ}}$ for P=0,1,2 is evaluated later in this Appendix.

For N+P+Q+1 = 0, equation A.2.7 rewritten as

$$z_{A2NPQ} = \frac{1}{\Lambda_2 R_2} z_{A2(N+2)(P-4)(Q+2)} - \left[\frac{R_1}{R_2} + \frac{\Lambda_1}{\Lambda_2}\right] z_{A2N(P-1)Q}$$

$$-\left[\frac{R_0}{R_2} + \frac{\Lambda_1 R_1}{\Lambda_2 R_2} + \frac{\Lambda_0}{\Lambda_2}\right] z_{A2N(P-2)Q} - \left[\frac{\Lambda_1 R_0}{\Lambda_2 R_2} + \frac{\Lambda_0 R_1}{\Lambda_2 R_2}\right] z_{A2N(P-3)Q}$$

$$-\frac{\Lambda_0^R_0}{\Lambda_2^R_2} Z_{A2N(P-4)Q}$$
 (A.2.10)

can be used recursively until P is 0 or 1 or 2 or 3. However, for the given range of values, this equation need never be invoked.

Special Case

If $^{\Lambda}_{2}R_{2}^{}$ << 1, equation A.2.9 is very error sensitive. Equation A.2.8 rewritten by changing N to (N+2) and Q to (Q+2) as

$$z_{\text{A2NPQ}} = \frac{(\Lambda(1))^{\frac{1}{2}N+1} (R(1))^{\frac{1}{2}Q+1}}{(P+1)\Lambda_0 R_0} - \frac{(N+P+Q+5)\Lambda_2 R_2}{(P+1)\Lambda_0 R_0} z_{\text{A2N}(P+4)Q}$$

$$-\frac{1}{P+1}\left[\frac{\Lambda_{1}R_{2}}{\Lambda_{0}R_{0}}\left(\frac{1}{2}N+P+Q+4\right) + \frac{\Lambda_{2}R_{1}}{\Lambda_{0}R_{0}}\left(N+P+\frac{1}{2}Q+4\right)\right] z_{A2N(P+3)Q}$$

$$-\frac{1}{P+1}\left[\frac{R_2}{R_0}(P+Q+3) + \frac{\Lambda_1R_1}{\Lambda_0R_0}(\frac{1}{2}N+P+\frac{1}{2}Q+3) + \frac{\Lambda_2}{\Lambda_0}(N+P+3)\right]z_{A2N(P+2)Q}$$

$$-\frac{1}{P+1}\left\{\frac{R_{1}}{R_{0}}\left(P+\frac{1}{2}Q+2\right)+\frac{\Lambda_{1}}{\Lambda_{0}}\left(\frac{1}{2}N+P+2\right)\right\} z_{A2N(P+1)Q}$$
(A.2.11)

along with the approximation that for $^{\Lambda}_{2}R_{2}$ << 1,

$$Z_{A2NPQ} = \frac{1}{P+1} (\Lambda(1))^{\frac{1}{2}(N+2)} (R(1))^{\frac{1}{2}(Q+2)}$$
 (A.2.12)

is used for the evaluation of $Z_{\mbox{A2NPQ}}(..)$ for all N, P, Q.

End of Special Case

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 Z_{A2NPO} is evaluated for P = 0,1,2 by writing

$$Z_{A2NPQ} = \Lambda_2^{\frac{1}{2}N} R_2^{\frac{1}{2}Q} \int_0^1 \left(\frac{1}{\Lambda_2} + \frac{\Lambda_1}{\Lambda_2} x + x^2 \right)^{\frac{1}{2}N} x^P \left(\frac{1}{R_2} + \frac{R_1}{R_2} x + x^2 \right)^{\frac{1}{2}Q} dx$$
(A.2.13)

If $\frac{\Lambda_1}{\Lambda_2} = \frac{R_1}{R_2}$, the substitution $t = 2x + \frac{R_1}{R_2}$ is used to give

$$z_{A2NPQ} = \frac{\Lambda_2^{\frac{1}{2}N} R_2^{\frac{1}{2}Q}}{2^{N+P+Q+1}} \int_{\frac{R_1}{R_2}}^{2+\frac{R_1}{R_2}} \left[t^2 + \frac{4\Lambda_2^{-\Lambda_1^2}}{\Lambda_2^2} \right]^{\frac{1}{2}N} \left[t - \frac{R_1}{R_2} \right]^p \left[t^2 + \frac{4R_2^{-R_1^2}}{R_2^2} \right]^{\frac{1}{2}Q} dt$$

$$= \frac{\Lambda_2^{\frac{1}{2}N}R_2^{\frac{1}{2}Q}}{2^{N+P+Q+1}} \stackrel{P}{\underset{s=0}{\Sigma}} \left(\stackrel{P}{s} \right) \left(-\frac{R_1}{R_2} \right)^{P-s} Z_{A3NsQ} \left(\frac{R_1}{R_2}, 2 + \frac{R_1}{R_2}, \Delta_{\Lambda}, \Delta_{R} \right)$$

(A.2.14)

where
$$Z_{A3NsQ}$$
 $(t_0, t_1, \Delta_1, \Delta_2) = \int_{t_0}^{t_1} (t^2 + \Delta_1)^{t_2N} t^S (t^2 + \Delta_2)^{t_2Q}$ dt (A.2.15) and where $\Delta_{\Lambda} = \frac{4\Lambda_2 - \Lambda_1^2}{\Lambda_2^2}$; $\Delta_{R} = \frac{4R_2 - R_1^2}{R_2^2}$. If $\frac{\Lambda_1}{\Lambda_2} \neq \frac{R_1}{R_2}$, the substitution

$$x = a + bB(t)$$

where $B(t) = \frac{t-1}{t+1}$ is the bilinear transformation on t and

where

$$a = \frac{\Lambda_2 - R_2}{\Lambda_1 R_2 - \Lambda_2 R_1}; b = \left[a^2 - \frac{R_1 - \Lambda_1}{\Lambda_1 R_2 - \Lambda_2 R_1}\right]^{\frac{1}{2}} = \left[a^2 + \frac{1}{\Lambda_2} + a\frac{\Lambda_1}{\Lambda_2}\right]^{\frac{1}{2}} = \left[a^2 + \frac{1}{R_2} + a\frac{R_1}{R_2}\right]^{\frac{1}{2}}$$
(A.2.16)

The above substitution* gives

*Noting that

$$t \approx B \left(\frac{b}{a-x}\right)$$
; $dx = \frac{2b}{(t+1)^2} dt$;

The first factor becomes

$$\frac{(2b)^{\frac{1}{2}N} \left(a+b+\frac{\Lambda_{1}}{2\Lambda_{2}}\right)^{\frac{1}{2}N}}{(t+1)^{N}} \qquad \left| t^{2} + B \left(\frac{b}{a+\frac{\Lambda_{1}}{2\Lambda_{2}}}\right) \right|^{\frac{1}{2}N}$$

The second factor

$$\frac{\left(a+b\right)^{P}}{\left(t+1\right)^{P}} \quad \left| t + B \left(\frac{\underline{a}}{b}\right) \right|^{P} = \left(a+b\right)^{P} \quad \sum_{s=0}^{P} \left(s\right)^{s} \left[B \left(\frac{\underline{a}}{b}\right) - 1 \right]^{P-s} \quad (t+1)^{s-P}$$

The third factor becomes

$$\frac{(2b)^{\frac{1}{2}Q} \left(a+b+\frac{R_1}{2R_2}\right)^{\frac{1}{2}Q}}{(t+1)^{Q}} \left| t^2 + B \left(\frac{b}{a+\frac{R_1}{2R_2}}\right) \right|^{\frac{1}{2}Q}$$

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$$Z_{A2NPQ} = \Lambda_{2}^{\frac{1}{2}N} R_{2}^{\frac{1}{2}Q} (2b)^{\frac{1}{2}N + \frac{1}{2}Q + 1} \left(a + b + \frac{\Lambda_{1}}{2\Lambda_{2}} \right)^{\frac{1}{2}N} \left(a + b + \frac{R_{1}}{2R_{2}} \right)^{\frac{1}{2}Q} (a + b)^{P}$$

$$\times \sum_{s=0}^{P} \left[{p \choose s} \left({a \choose b} - 1 \right)^{P-s} Z_{A4N(s-N-P-Q-2)Q} \left({a \choose a}, {a \choose a-1}, {a \choose a+\frac{\Lambda_{1}}{2\Lambda_{2}}} \right)^{p} \left({a \choose a+\frac{\Lambda_{1}}{2R_{2}}} \right)^{p} \left({a \choose a+2,17} \right)^{p} \left({a \choose$$

where

$$z_{A4NPQ}$$
 $(t_0, t_1, \Delta_1, \Delta_2) = \begin{cases} t_1 \\ t_0 \end{cases}$ $(t^2 + \Delta_1)^{\frac{1}{2}N}$ $(t+1)^P$ $(t^2 + \Delta_2)^{\frac{1}{2}Q}$ dt (A.2.18)

Beginning of $Z_{A2(-1)P(-1)}$ evaluation

 Z A2(-1)P(-1) is evaluated using equation(A.2.14) if $\frac{\Lambda_{1}}{\Lambda_{2}} = \frac{R_{1}}{R_{2}}$ and equation (A.2.17) if $\frac{\Lambda_{1}}{\Lambda_{2}} \neq \frac{R_{1}}{R_{2}}$, or equation (A.2.10) if P=1

End of $Z_{A2(-1)P(-1)}$ evaluation

- Marie Mari

Beginning of $Z_{A2(-2)P(-1)}$ evaluation

Beginning of $Z_{A2(-2)2(-1)}$ evaluation

$$Z_{A2(-2)2(-1)} = \frac{1}{\Lambda_2 R_2^{\frac{1}{2}}} \int_0^1 \frac{x^2}{\left(\frac{1}{\Lambda_2} + \frac{\Lambda_1}{\Lambda_2} x + x^2\right) \left(\frac{1}{R_2} + \frac{R_1}{R_2} x + x^2\right)^{\frac{1}{2}}} dx$$

$$= \frac{1}{\Lambda_2 R_2^{\frac{1}{2}}} \int_0^1 \frac{1}{\left(\frac{1}{R_2} + \frac{R_1}{R_2} x + x^2\right)^{\frac{1}{2}}} dx$$

$$-\frac{\Lambda_1}{\Lambda_2} Z_{A2(-2)1(-1)} - \frac{1}{\Lambda_2} Z_{A2(-2)0(-1)}$$

or

$$z_{A2(-2)2(-1)} = \frac{1}{\Lambda_2} z_{A10(-1)}([R]) - \frac{\Lambda_1}{\Lambda_2} z_{A2(-2)1(-1)} - \frac{1}{\Lambda_2} z_{A2(-2)0(-1)}$$

End of $Z_{A2(-2)2(-1)}$ evaluation

 $Z_{A2(-2)P(-1)}$ for P=0 or 1 is evaluated using equation (A.2.14) if

$$\frac{\Lambda_1}{\Lambda_2} = \frac{R_1}{R_2} \quad \text{and (A.2.17) if } \quad \frac{\Lambda_1}{\Lambda_2} \neq \frac{R_1}{R_2}.$$

End of $Z_{A2(-2)P(-1)}$ evaluation

A.3 The Evaluation Procedure for z_{A3NPQ}

$$z_{A3NPQ} = \int_{t_0}^{t_1} (t^2 + \Delta_1)^{\frac{1}{2}N} t^P (t^2 + \Delta_2)^{\frac{1}{2}Q} dt$$

Beginning of $z_{A3(-1)P(-1)}$ evaluation

If $\alpha_2 < \alpha_1$, we write

$$z_{A3(-1)P(-1)}$$
 $(t_0,t_1,\alpha_1,\alpha_2) = z_{A3(-1)P(-1)}$ $(t_0,t_1,\alpha_2,\alpha_1)$

From this point on it is assumed that $\alpha_2 \ge \alpha_1$

Let
$$t = \alpha_1^{\frac{1}{2}} \tan \theta$$
; $\theta_1 = \arctan \alpha_1^{-\frac{1}{2}} t_1$; $\theta_0 = \arctan \alpha_1^{-\frac{1}{2}} t_0$

$$Z_{A3(-1)P(-1)} = \int_{\theta_0}^{\theta_1} \frac{\alpha_1^{\frac{1}{2}(P)} (\tan \theta)^P \sec^2 \theta \alpha_1^{\frac{1}{2}} d\theta}{(\alpha_1 \alpha_2)^{\frac{1}{2}} \sec^2 \theta (\frac{\alpha_1}{\alpha_2} \sin^2 \theta + \cos^2 \theta)^{\frac{1}{2}}}$$

$$= \frac{\alpha_1^{\frac{1}{2}(P)}}{\alpha_2^{\frac{1}{2}}} \int_{\theta_0}^{\theta_1} \frac{\tan^P \theta \ d\theta}{\left(1 - \left(1 - \frac{\alpha_1}{\alpha_2}\right) \sin^2\theta\right)^{\frac{1}{2}}}$$

Letting
$$\Delta(\theta) = \sqrt{1 - \left(1 - \frac{\alpha_1}{\alpha_2}\right) \sin^2 \theta}$$
, $k^2 = 1 - \frac{\alpha_1}{\alpha_2}$, $k^{-2} = \frac{\alpha_1}{\alpha_2}$

$$Z_{A3}^{(-1)0(-1)} = \frac{1}{(\alpha_2)^{\frac{1}{2}}} \left| F(\theta_1, k) - F(\theta_0, k) \right| \quad (G \in \mathbb{R} \ 2.584.1)$$

$$Z_{A3(-1)1(-1)} = \left[\frac{\alpha_1}{\alpha_2}\right]^{\frac{1}{2}} \frac{1}{2k} \left| \ln \left[\frac{\Delta(\theta_1) + k'}{\Delta(\theta_1) - k'} \frac{\Delta(\theta_0) - k'}{\Delta(\theta_0) + k'}\right] \right| (G \& R 2.584.73)$$

$$= \frac{1}{2} \ln \left[\frac{\Delta(\theta_1) + k'}{\Delta(\theta_0) + k'} \frac{\Delta(\theta_0) - k'}{\Delta(\theta_1) - k'}\right]$$

$$Z_{A3(-1)2(-1)} = \frac{\alpha_1}{\alpha_2^{\frac{1}{2}} k'^2} \left| \Delta(\theta_1) \tan \theta_1 - E(\theta_1, k) - \Delta(\theta_0) \tan \theta_0 \right| + E(\theta_0, k)$$

$$= \left(\omega_{2} \right)^{\frac{1}{2}} \begin{vmatrix} \Delta(\theta_{1}) \tan \theta_{1} - \Delta(\theta_{0}) \tan \theta_{0} \\ -E(\theta_{1}, k) + E(\theta_{0}, k) \end{vmatrix}$$

Where $F(\theta,k)$ and $E(\theta,k)$ are elliptic integrals.

When $\alpha_1 = \alpha_2$, these simplify to

$$Z_{A3(-1)0(-1)} = \frac{1}{(\alpha_2)^{\frac{1}{2}}} (\theta_1 - \theta_0)$$

$$Z_{A3(-1)1(-1)} = \frac{1}{2} \log \frac{t_1^2 + \alpha_1}{t_0^2 + \alpha_1}$$

$$Z_{A3(-1)2(-1)} = (t_1 - t_0) - \alpha_1 Z_{A3(-1)0(-1)}$$

End of $Z_{A3(-1)P(-1)}$ Evaluation

(G&R 2.584.90)

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Beginning of $Z_{A3(-2)P(-1)}$ Evaluation

 $Z_{A3(-2)P(-1)}$ is evaluated using the substitution $t = \alpha_2^{\frac{1}{2}} \tan \theta$

$$z_{A3(-2)P(-1)} = \alpha_2^{\frac{1}{2}P} \int_{\theta_0}^{\theta_1} \frac{\cos\theta \tan^P\theta d\theta}{\alpha_1 \cos^2\theta + \alpha_2 \sin^2\theta}$$

where $\theta_0 = \arctan \alpha_2^{-\frac{1}{2}} t_0$; $\theta_1 = \arctan \alpha_2^{-\frac{1}{2}} t_1$.

For P = 0, the substitution $u = \sin\theta$ gives

$$z_{A3(-2)0(-1)} = z_{A3} (\sin \theta_0, \sin \theta_1, \alpha_2 - \alpha_1, \alpha_1)$$

where

$$\begin{split} \mathbf{Z}_{\mathbf{A}3} & \left(\mathbf{u}_{0}, \ \mathbf{u}_{1}, \ \mathbf{b}, \ \mathbf{a}\right) = \int_{\mathbf{u}_{0}}^{\mathbf{u}_{1}} \frac{d\mathbf{u}}{\mathbf{a} + \mathbf{b}\mathbf{u}^{2}} \\ & = \frac{1}{\sqrt{\mathbf{a}\mathbf{b}}} \left\{ \operatorname{arctan} \left(\left(\frac{\mathbf{b}}{\mathbf{a}}\right)^{\frac{1}{2}} \mathbf{u}_{1} \right) - \operatorname{arctan} \left(\left(\frac{\mathbf{b}}{\mathbf{a}}\right)^{\frac{1}{2}} \mathbf{u}_{0} \right) \right\} & \text{if } \mathbf{a} \mathbf{b} > 0, \\ & = \frac{1}{\mathbf{a}} \left(\mathbf{u}_{1} - \mathbf{u}_{0} \right) & \text{if } \mathbf{b} = 0 \\ & = \frac{1}{\mathbf{b}} \frac{\mathbf{u}_{1} - \mathbf{u}_{0}}{\mathbf{u}_{0} \mathbf{u}_{1}} & \text{if } \mathbf{a} = 0 \\ & = \frac{1}{2 \left(-\mathbf{a}\mathbf{b} \right)^{\frac{1}{2}}} & \text{in } \left(\frac{\mathbf{a} + \left(-\mathbf{a}\mathbf{b} \right)^{\frac{1}{2}} \mathbf{u}_{1}}{\mathbf{a} + \left(-\mathbf{a}\mathbf{b} \right)^{\frac{1}{2}} \mathbf{u}_{0}} \frac{\mathbf{a} - \left(-\mathbf{a}\mathbf{b} \right)^{\frac{1}{2}} \mathbf{u}_{0}}{\mathbf{a} - \left(-\mathbf{a}\mathbf{b} \right)^{\frac{1}{2}} \mathbf{u}_{1}} \right) & \text{if } \mathbf{a} \mathbf{b} < 0 \end{split}$$

For P = 1, the substitution $u = \cos \theta$ gives

$$z_{A3(-2)1(-1)} = -\alpha_2^{\frac{1}{2}} z_{A3} \left[\cos\theta_0, \cos\theta_1, (\alpha_1 - \alpha_2), \alpha_2 \right]$$

End of Z_{A3(-2)P(-1)} Evaluation

A.4 The Evaluation Procedure for Z_{A4NPQ}

A procedure for evaluating

$$Z_{A4NPQ}(t_0, t_1, \alpha_1, \alpha_2) = \int_{t_0}^{t_1} (t^2 + \alpha_1)^{\frac{1}{2}N} (t+1)^P (t^2 + \alpha_2)^{\frac{1}{2}Q} dt$$

is discussed in this section for P = 1, 0, -1, -2. If P = 0

$$z_{A4N0Q}$$
 (....) = z_{A3N0Q} (....)

If P = 1,

$$z_{A4N1Q}(\ldots) = z_{A3N0Q}(\ldots) + z_{A3N1Q}(\ldots)$$

$$\begin{split} &\mathbf{Z}_{\mathbf{A4NPQ}}(\dots) &\text{ for negative values of } \mathbf{P} &\text{ is invoked only when } \\ &\mathbf{N}=-1, \ \mathbf{Q}=-1. &\text{ In that case, if } \alpha_2 < \alpha_1, \ \text{we write} \\ &\mathbf{Z}_{\mathbf{A4}\,(-1)\,\mathbf{P}\,(-1)} & \left(\mathbf{t}_0\,,\mathbf{t}_1\,,\alpha_1\,,\alpha_2\right) = \mathbf{Z}_{\mathbf{A4}\,(-1)\,\mathbf{P}\,(-1)} & \left(\mathbf{t}_0\,,\mathbf{t}_1\,,\alpha_2\,,\alpha_1\right). \\ &\text{From this point on it is assumed that } \alpha_2 \geq \alpha_1. &\text{Let} \\ &\mathbf{t}=\alpha_1^{\frac{1}{2}} \tan\theta; \quad \mathbf{t}_0=\alpha_1^{\frac{1}{2}} \tan\theta_0; \quad \mathbf{t}_1=\alpha_1^{\frac{1}{2}} \tan\theta_1. \end{split}$$

$$Z_{A4(-1)P(-1)} = \frac{1}{\alpha_2^{\frac{1}{2}}} \int_{\theta_0}^{\theta_1} \frac{\left(1 + \alpha_1^{\frac{1}{2}} \tan \theta\right)^P d\theta}{\left[1 - \left(1 - \frac{\alpha_1}{\alpha_2}\right) \sin^2 \theta\right]^{\frac{1}{2}}}$$

Letting
$$\Delta(\theta) = \sqrt{1 - k^2 \sin^2 \theta}$$
 where $k^2 = 1 - \frac{\alpha_1}{\alpha_2}$, $k^2 = \frac{\alpha_1}{\alpha_2}$,

we write

$$Z_{A4(-1)P(-1)} = \frac{\alpha_1^{\frac{1}{2}P}}{\alpha_2^{\frac{1}{2}}} \int_{\theta_0}^{\theta_1} \frac{(c + \tan\theta)^P d\theta}{\Delta(\theta)}$$

where $c = \alpha_1^{-\frac{1}{2}}$

$$Z_{A4(-1)(-1)(-1)} = \frac{1}{(1+\alpha_1)\alpha_2^{\frac{1}{2}}} \left[F(\theta_1,k) - F(\theta_0,k)\right]$$

$$+ \frac{\alpha_{1}}{\left(1 + \alpha_{1}\right) \alpha_{2}^{\frac{1}{2}}} \left[\pi \left(\theta_{1}, -(1 + \alpha_{1}), k\right) - \pi \left(\theta_{0}, -(1 + \alpha_{1}), k\right) \right]$$

$$-\frac{1}{2\sqrt{(1+\alpha_1)(1+\alpha_2)}} \ln \left[\frac{(1+\frac{1}{\alpha_2})^{\frac{1}{2}} + (1+\frac{1}{\alpha_1})^{\frac{1}{2}} \Delta(\theta_1)}{(1+\frac{1}{\alpha_2})^{\frac{1}{2}} - (1+\frac{1}{\alpha_1})^{\frac{1}{2}} \Delta(\theta_1)} - \frac{(1+\frac{1}{\alpha_2})^{\frac{1}{2}} - (1+\frac{1}{\alpha_1})^{\frac{1}{2}} \Delta(\theta_0)}{(1+\frac{1}{\alpha_2})^{\frac{1}{2}} + (1+\frac{1}{\alpha_1})^{\frac{1}{2}} \Delta(\theta_0)} \right]$$

Also from tables of integrals, (G&R 2.591.2)

$$\begin{split} Z_{\text{A4}(-1)(-2)(-1)} &= \frac{1}{\alpha_{1}\alpha_{2}^{\frac{1}{2}}} \left[\frac{\Delta (\theta_{1})}{(c + \tan \theta_{1}) \cos^{2}\theta_{1}} + \frac{\Delta (\theta_{0})}{(c + \tan \theta_{0}) \cos^{2}\theta_{0}} \right] \frac{\alpha_{1}\alpha_{2}}{(1 + \alpha_{1})(1 + \alpha_{2})} \\ &+ \frac{\alpha_{2}^{\frac{1}{2}}}{(1 + \alpha_{1})(1 + \alpha_{2})} \left[\frac{1}{\alpha_{1}^{\frac{1}{2}}} \left(1 + \frac{\alpha_{1}}{\alpha_{2}} + \frac{2}{\alpha_{2}} \right) Z_{\text{A4}(-1)(-1)(-1)} (\alpha_{1}\alpha_{2})^{\frac{1}{2}} \right] \\ &+ \tan \theta_{1} \Delta (\theta_{1}) - \tan \theta_{0} \Delta (\theta_{0}) \\ &- \frac{1}{\alpha_{2}} F(\theta_{1}, \mathbf{k}) + \frac{1}{\alpha_{2}} F(\theta_{0}, \mathbf{k}) \\ &- E(\theta_{1}, \mathbf{k}) + E(\theta_{0}, \mathbf{k}) \right] \end{split}$$

End of $Z_{A4(-1)(-2)(-1)}$ Evaluation

The elliptic integrals used in this evaluation are computed using standard routines from the SSP library.

A.5 The Solution For Z_{A5NQMS}(...)

$$Z_{A5NQMS}$$
 ([A], [B],[C]) = $\int_{0}^{1} (A(x))^{\frac{1}{2}N} (B(x))^{\frac{1}{2}Q} x^{M} (C(x))^{\frac{1}{2}S} dx$
(A.5.1)

is invoked only for N=0 or 2. If N=0,

$$Z_{A50QMS}$$
 (...) = Z_{A2QMS} ([B], [C]). (A.5.2)

If N=3

$$z_{A52QMS}$$
 (...) = $\sum_{i=0}^{2} A_i z_{A2Q(M+i)S}$ ([8],[c]) (A.5.3)

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APPENDIX B

CLOSED FORM SOLUTION FOR SOME DOUBLE INTEGRALS ARISING IN
THE COMPUTATION OF MUTUAL IMPEDANCE QUANTITIES

Notation:

$$R = R(x,y) = R_{x}(y) = R_{y}(x) = \sum_{i=0}^{2} \sum_{j=0}^{2-i} R_{ij} x^{i} y^{j}$$

Writing R(x,y), as the trinomial $R_{\hat{V}}(x)$ implies the form

$$R_{y}(x) = \sum_{i=0}^{2} \begin{bmatrix} 2-i \\ \sum \\ j=0 \end{bmatrix} R_{ij} y^{j}$$

Writing R(x,y) as the trinomial $R_{x}(y)$ implies the form

$$R_{\mathbf{x}}(\mathbf{y}) = \sum_{j=0}^{2} \begin{pmatrix} 2-j \\ j \end{pmatrix} R_{ij} \mathbf{x}^{i}$$
 \mathbf{y}^{j}

[R_x] is the vector
$$((R_{00} + R_{10}x + R_{20}x^2)(R_{01} + R_{11}x) R_{02})$$

Similar notation is used with $\Lambda(x,y)$. Furthermore, $[R_{x=0}]$ represents the vector $(R_{00} \ R_{01} \ R_{02})$ whereas $[R_{y=0}]$ represents the vector $(R_{00} \ R_{10} \ R_{20})$, etc. The trinomial $R_{x=0}(y)$ is written $R_0(y)$ for brevity.

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B.1 The Computation of $\mathbf{Z}_{\texttt{B1PQMS}}$

By definition,

$$z_{B1PQMS} = \int_{0}^{1} y^{M} (R_{0}(y))^{\frac{1}{2}S} \int_{0}^{1} x^{P} (R(x,y))^{\frac{1}{2}Q} dx dy$$
(B.1.1)

Writing $(R)^n = (R)^{n-2} R^2$ it is easily proved that

$$z_{B1PQMS} = \sum_{i=0}^{2} \sum_{j=0}^{2-i} R_{ij} z_{B1(P+i)(Q-2)(M+j)S}$$
 (B.1.2)

and

$$z_{B1PQMS} = \sum_{j=0}^{2} R_{0j} z_{B1 PQ (M+j) (S-2)}$$
 (B.1.3)

Equation B.1.2 is used recursively until $\,Q\,$ is zero or negative. Equation B.1.3 is used recursively until $\,S\,$ is zero or -1. A recursion relation for reducing the value of $\,M\,$ and $\,P\,$ is derived as follows:

Integrating the inner integral in equation B.1.1 by parts,

$$z_{B1PQMS} = \frac{1}{P+1} x^{P+1} \int_{0}^{1} y^{M} (R_{0}(y))^{\frac{1}{2}S} (R_{y}(x))^{\frac{1}{2}Q} dy \bigg]_{x=0}^{1}$$

$$-\frac{1}{P+1}\sum_{i=0}^{2}\sum_{j=0}^{2-i}\sum_{i=1}^{2}i_{i}Q_{i}R_{ij}Z_{B1(P+i)(Q-2)(M+j)}S$$
 (B.1.4)

Subtracting equation B.1.2 from B.1.4

In equation B.1.5, the sum of the subscripts in the P and M positions of $Z_{\text{Bl...}}$ is (P+M+i+j). This is highest when i+j=2. Separating all terms that meet this criterion

or changing P to P-2, Q to Q+2,

$$\sum_{i=0}^{2} (P + \frac{1}{2} i Q + i - 1) R_{i}(2-i) Z_{B1}(P - (2-i)) Q(M+2-i) S$$

$$= x^{P-1} Z_{A2}(Q+2) MS(...) \Big]_{x=0}^{1}$$

$$- \sum_{i=0}^{2} \sum_{j=0}^{2} (P + \frac{1}{2} i Q + i - 1) R_{ij} Z_{B1}(P - (2-i)) Q(M+j) S \qquad (B.1.6)$$

. .

The implications of this recurrence relation are shown in Figure B.l.l. For a given value of Q and S, different Z_{B1PQMS} values map onto a grid in the P-M plane. The points of the grid with two circles around them (A and G) are the given values of P_0 and M_0 . When $P_0 \neq 1$ (Point A) equation B.l.6 gives a linear combination of Points A, B and C in terms of D, E and F. When $P_0 = 1$ (Point G), a linear combination of points G and H is given in terms of Point J. In the grid in the P-M plane as shown in Figure B.l.2, it is assumed that all $Z_{B1...}$ for which $P + M \leq n - 1$ are known. Let $Z_{Pn} \stackrel{\triangle}{=} Z_{B1PQ(n-P)S}$, where all Z_{Pn} (P = 0 to n) are to be determined. The application of equation B.l.6 for all Z_{Pn} (P = 1 to n) gives n linear equations to be solved simultaneously. Another equation is generated as follows:

Integrating B.1.1 by parts,

$$Z_{B1PQMS} = \frac{1}{M+1} \left\{ y^{M+1} \left(R_{0}(y) \right)^{\frac{1}{2}S} Z_{A1PQ} \left(\left[R_{X}(y) \right] \right) \right\}_{y=0}^{1}$$

$$- \sum_{i=0}^{2} \sum_{j=0}^{2-i} \sum_{i=0}^{2} Q_{j} R_{ij} Z_{B1(P+i)(Q-2)(M+j)S}$$

$$- \sum_{j=0}^{2} \sum_{i=0}^{2} S_{i} R_{0j} Z_{B1PQ(M+j)(S-2)} \right\}$$
(B.1.7)

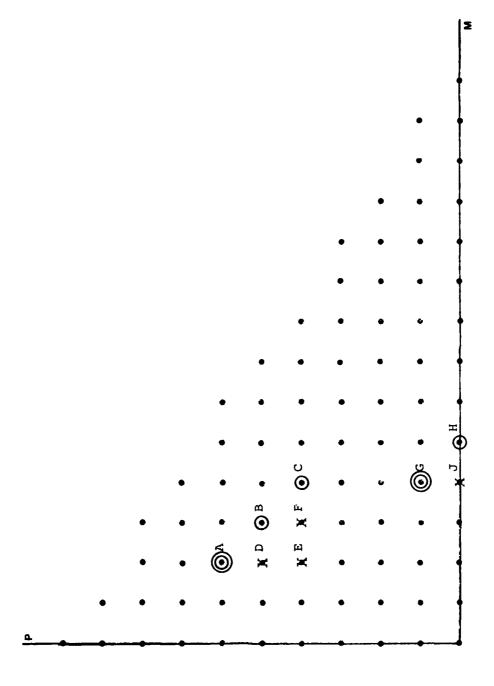


Figure B.1.1

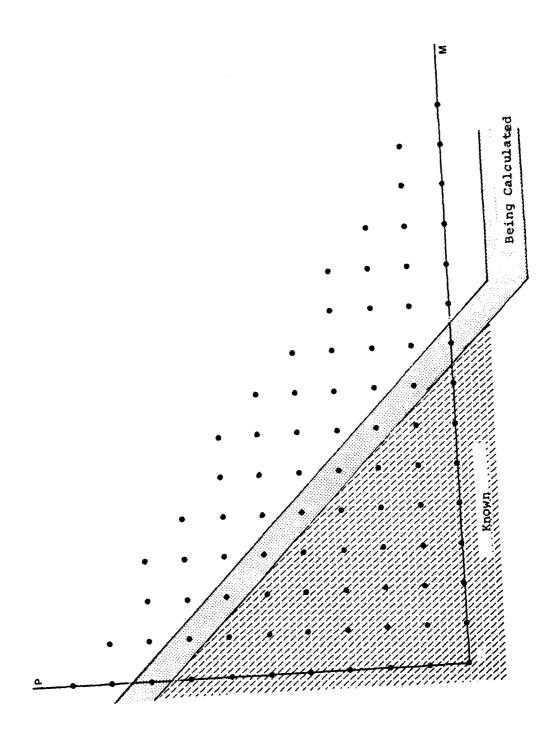


Figure B.1.2

Substituting equation B.1.2 into the last term and equation B.1.3 into the second term, equation B.1.7 becomes

$$Z_{B1PQMS} = \frac{1}{M+1} y^{M+1} (R_0(y))^{\frac{1}{2}S} Z_{A1PQ} ([R_X(y)]) \Big]_{y=0}^{1}$$

$$- \frac{1}{M+1} \sum_{j=0}^{2} \sum_{i=0}^{2} \sum_{k=0}^{2} (\frac{1}{2} jQ + \frac{1}{2} kS) R_{ij} R_{0k}$$

$$\times Z_{B1(P+i)(Q-2)(M+j+k)(S-2)}$$
(B.1.8)

Combining equations B.1.2 and B.1.3, another equation for $$^{\rm Z}_{\rm B1POMS}$$ is obtained.

Subtracting B.1.9 from B.1.8 and multiplying throughout by (M+1),

$$0 = y^{M+1} (R_0(y))^{\frac{1}{2}S} Z_{AlPQ} ([R_X(y)]) \bigg]_{y=0}^{1}$$

$$- \sum_{j=0}^{2} \sum_{i=0}^{2} \sum_{k=0}^{2} (\frac{1}{2} jQ + M+1 + \frac{1}{2} kS) R_{ij}^{R_0k} Z_{Bl(P+i)(Q-2)(M+j+k)(S-2)}$$

(B.1.10)

In equation B.1.10, the sum of the subscripts in the P and M positions $Z_{\text{Bl...}}$ is (P+M+i+j+k). This is highest when k=2 and i+j=2. Separating all terms that meet this criterion,

$$\sum_{j=0}^{2} (\frac{1}{2} jQ + M + 1 + S) R_{(2-j)j} R_{02} Z_{B1(P+2-j)(Q-2)(M+2+j)(S-2)}$$

$$= y^{M+1} (R_{0}(y))^{\frac{1}{2}S} Z_{A1PQ} (\cdot) \Big]_{y=0}^{1}$$

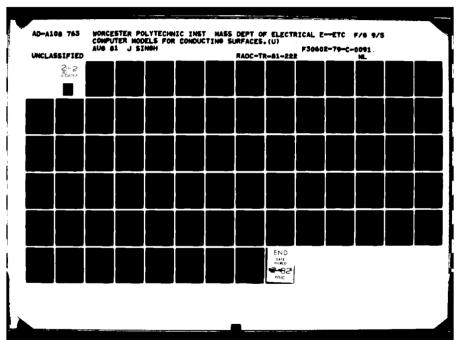
$$- \sum_{j=0}^{2} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (\frac{1}{2} jQ + M + 1 + \frac{1}{2} kS) R_{ij} R_{0k}$$

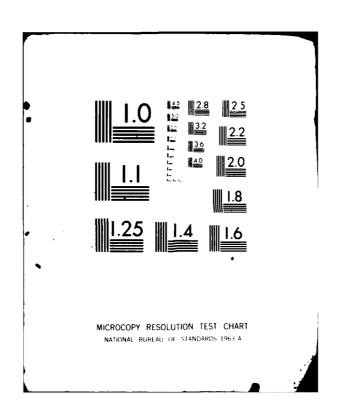
 $x = {}^{2}B1(P+i)(Q-2)(M+j+k)$ (S-2)

or changing Q to Q+2, M to M-4, S to S+2,

$$\begin{array}{c} 2 \\ \Sigma \\ j=0 \end{array} (^{1}2^{j}Q + M + S - 1 + j) & R_{(2-j)j} & R_{02} & ^{2}B1 \big(P + 2 - j \big) Q \big(M - 2 + j \big) & S \\ \\ = y^{M-3} & \big(R_{0}(y) \big)^{\frac{1}{2}} (S + 2) & Z_{A1(P)(Q + 2)} & (\cdot) \bigg]_{Y=0}^{1} \\ \\ - & \Sigma & \Sigma & \Sigma & \Sigma \\ & j=0 & i=0 & k=0 \end{array} (^{1}2^{j}Q + M + ^{1}2^{j}kS - 3 + j + k) & R_{ij} & R_{0k} \\ \\ & & ^{2}B1(P + i)Q(M - 4 + j + k) & S \end{array}$$

The implications of this recursion relation are shown in Figure B.1.3. For a given value of Q and S, different z_{BlPOMS} values map onto a grid in the PM plane. The point with the circle on it is a given value of P_0 and M_0 . The left-hand side of equation B.1.11 is a linear combination of the points in the shading. The right-hand side maps the points in the dashed Thus equation B.1.11 calculates a linear combination area. $\mathbf{z}_{\mathrm{B1}...}$ quantities in terms of the quantities marked of the dotted with dashes (which have M ranging from $M_0 - 4$ to $M_0 - 1$, P from P_0 to $P_0 + 2$). It is also to be noted that the leftmost three points correspond to j = k = 0. Therefore, the multiplier for those points in equation B.L11 is $(M-3)R_{i0}R_{00}$. Thus when $M_0 = 3$, only the terms involving M from 0 to 2 are needed for the computation of the weighted sum of the dotted points. For the calculation of Z_{pn} , equation B.1.11 gives n-l linear equations, yielding a total of 2 n - 1 equations (n linear equations from B.1.6) to solve for the n+1 values of Z_{pn} The n+1 equations are chosen using n . equations resulting from letting P=n down to l(M=n-P) in equation B.1.6 and 1 equation resulting from letting M = n (P = 0) in equation B.1.11. This set has been chosen so as to maximize the use of equation B.1.6 which is simpler in form.





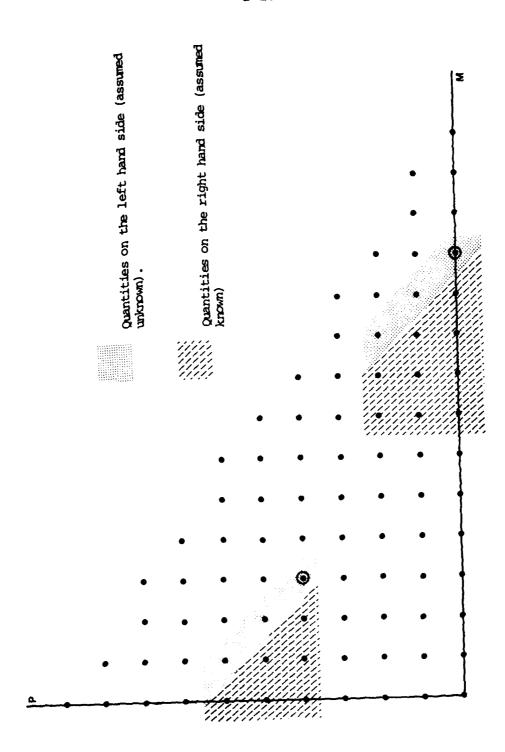


Figure B.1.3: A Pictorial Representation of Equation B.1.11

The application of equation B.1.11 for P = 0, M = n, gives

$$(n+s-1) R_{20} R_{02} Z_{2n} + (n+s-1+\frac{1}{2} (Q+2)) R_{11} R_{02} Z_{1n}$$

$$+ (n+s-1+(Q+2)) R_{02} R_{02} Z_{0n} = S_{1n}$$
(B.1.12a)

where

$$S_{ln} = y^{n-3} \left(R_0(y) \right)^{\frac{1}{2}(S+2)} Z_{Al0(Q+2)} \left(\left[R_X(y) \right] \right) \int_{y=0}^{1} Z_{Al0(Q+2)} \left(\left[R_X(y) \right] \right) dy$$

$$- \sum_{j=0}^{2} \sum_{i=0}^{2-j} \min_{k=0}^{\min(3-i-j,2)} (\frac{1}{2}jQ + n + \frac{1}{2}kS - 3 + j + k) R_{ij} R_{0k}$$

$$^{Z}Bl(0+i)Q(n-4+j+k) S (B.1.13a)$$

The application of equation B.1.6 for P=1 and 2 (M=n-1) and n-2 respectively) gives

$$\frac{1}{2} (Q+2) R_{11} Z_{0n} + (Q+2) R_{20} Z_{1n} = S_{2n}$$
 (B.1.12b)

$$R_{02} Z_{0n} + (1 + \frac{1}{2} (Q + 2)) R_{11} Z_{1n} + (1 + (Q + 2)) R_{20} Z_{2n} = S_{3n} (B.1.12c)$$
 where

$$s_{2n} = z_{A2(Q+2)(n-1)S}([R_y(1)],[R_y(0)]) - z_{Al(n-1)(Q+S+2)}([R_y(0)])$$

and where

$$s_{3n} = z_{A2(Q+2)(n-2)S} ([R_{y}(1)], [R_{y}(0)])$$

$$- \sum_{i=0}^{L} \sum_{j=0}^{\Sigma} (1 + \frac{1}{2} i(Q+2)) R_{ij} z_{B1 iQ(n+j-2)S}$$
(B.1.13c)

Equations B.1.12 are solved simultaneously to give z_{0n} , z_{1n} and z_{2n} . It is seen that the three equations are singular if Q=-2 in which case z_{0n} and z_{1n} are computed numerically and z_{2n} is computed using equation B.1.12a. Once z_{0n} , z_{1n} and z_{2n} are known, all other z_{pn} are computed by writing equation B.1.6 in the form

Figure B.1.4 summarizes the calculation strategy.

A recursion relation that is simpler than equation B.1.11 can be derived when S=0. Under this condition the last term in equation B.1.7 vanishes and, subtracting B.1.2 from B.1.7,

The same

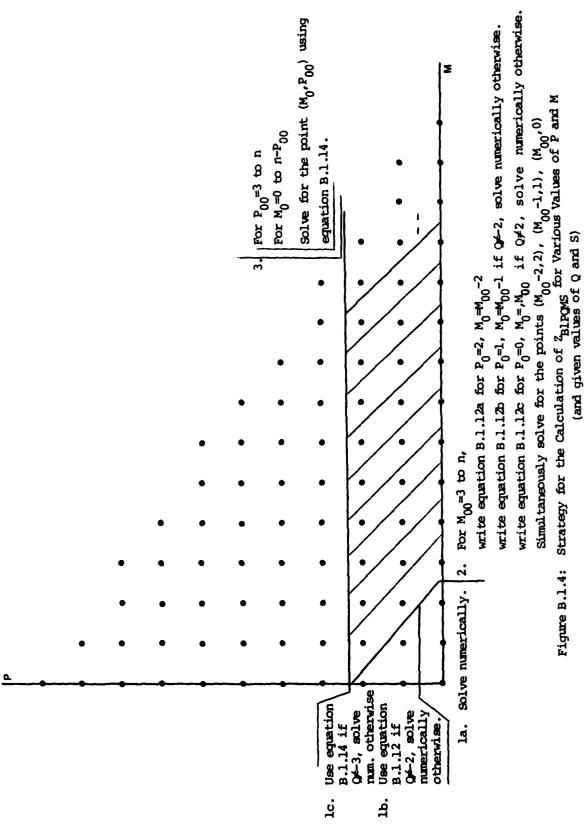


Figure B.1.4:

$$0 = y^{M+1} z_{AlpQ}([R_x(y)]) \Big]_{y=0}^{1}$$

$$- \sum_{i=0}^{2} \sum_{j=0}^{2-i} (l_{2}jQ + M + 1) R_{ij} z_{Bl(P+i)(Q-2)(M+j)0}$$
(B.1.15)

Separating all terms for which i + j = 2,

$$\begin{array}{l} 2 \\ \Sigma \\ j=0 \end{array} (\begin{array}{l} \frac{1}{2} \ jQ + M + 1) R_{(2-j)j} & Z_{B1(P+2-j)(Q-2)(M+j)0} \\ \\ = y^{M+1} \ Z_{A1PQ} \left(\left[R_{x}(y) \right] \right) \\ \\ - \begin{array}{l} \sum \\ \Sigma \\ i=0 \end{array} \sum _{j=0}^{\infty} \left(\begin{array}{l} \frac{1}{2} \ jQ + M + 1 \right) R_{ij} & Z_{B1(P+i)(Q-2)(M+j)0} \end{array}$$

or, replacing M by M-2, Q by Q+2 throughout,

Using B.1.6 and B.1.16, all points on the P-M grid except P=M=0 become candidates for recursion. When n=1, M=n, P=0, equation B.1.16 gives

$$\frac{1}{2}(Q+2)$$
 R_{11} z_{10}^{+} $(Q+2)$ R_{02} z_{01}^{-} = $z_{AlP(Q+2)}$ ([$R_{x}(1)$])

$$- z_{A1P(Q+2)}([R_X(0)]) - \frac{1}{2}(Q+2) R_{01} z_{B10Q00}$$
 (B.1.17)

which is solved simultaneously with equation B.1.12b to give z_{01} and z_{10} . When n=2, equation B.1.16 for M=2, P=0 becomes

$$R_{20} \quad z_{20} + (1 + \frac{1}{2}(Q+2)) \quad R_{11} \quad z_{11} + (1 + (Q+2)) \quad R_{02} \quad z_{02}$$

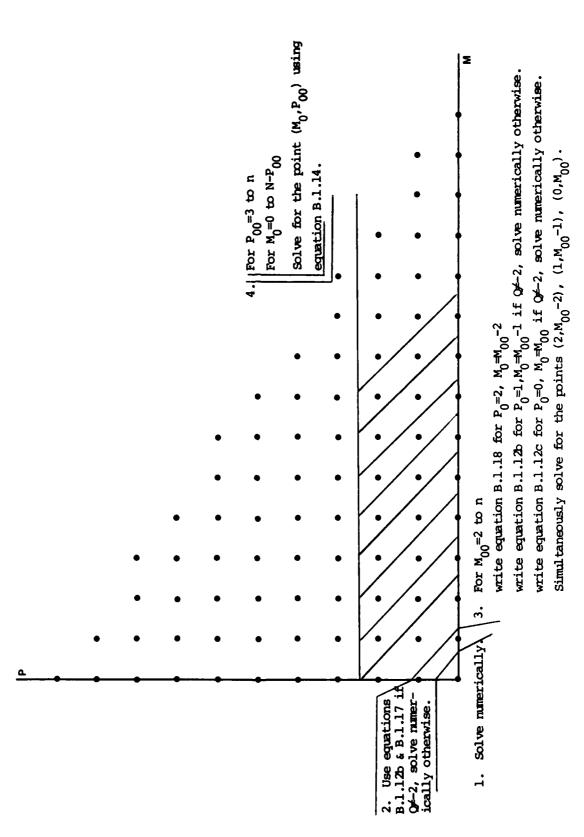
$$= z_{A10(Q+2)} \left\{ \begin{bmatrix} R_{x}(1) \end{bmatrix} \right\}$$

$$- \sum_{i=0}^{2} \sum_{j=0}^{2} (\frac{1}{2}j(Q+2) + 1) \quad R_{ij} \quad z_{B1iQj0}$$
(B.1.18)

Figure B.1.5 shows the P-M grid and outlines the stragety for calculating all $\rm \sc z_{B1PQM0}.$

The numerical calculations are performed using a 6 roint Gaussian integration (Abramowitz and Stegan, formulae 25.4.30) whereby

, ,



for Various Values of P & M (and given Q) B1PQM0 Figure B.1.5: Strategy for the Calculation of Z

$$\int_{0}^{1} f(y) dy = \frac{1}{2} \sum_{i=1}^{6} w_{i} f(\frac{1}{2}x_{i} + \frac{1}{2})$$

where w_i and x_i are given by (Abramowitz and Stegun, Table 25.4) for various values of n. Thus,

$$z_{\text{BlPQMS}} = \frac{1}{2} \sum_{i=1}^{6} w_i (y_i)^{M} (R_0(y_i))^{S/2} z_{\text{AlPQ}} ([R_{y=y_i}])$$
 where $y_i = \frac{1}{2} (x_i+1)$. (B.1.19)

B.2 Closed Form Solution of $Z_{\text{B2NPOMS}}(..)$.

By definition,

$$z_{B2NPQMS} = \int_{0}^{1} y^{M}(R_{0}(y))^{\frac{1}{2}S} \int_{0}^{1} (\Lambda(x,y))^{\frac{1}{2}N} x^{P} (R(x,y))^{\frac{1}{2}Q} dx dy$$
(B.2.1)

Writing $(.)^n = (.)^2 (.)^{n-2}$ it is easy to see that

$$z_{B2NPQMS} = \begin{cases} 2 & 2-i \\ \sum & \sum \\ i=0 & j=0 \end{cases} R_{ij} Z_{B2N(P+i)(Q-2)(M+j)S}$$
 (B.2.2)

$$z_{B2NPQMS} = \sum_{j=0}^{2} R_{0j} z_{B2NPQ(M+j)(S-2)}$$
 (B.2.3)

$$Z_{\text{B2NPQMS}} = \sum_{i=0}^{2} \sum_{j=0}^{2-i} \Lambda_{ij} Z_{\text{B2(N-2) (P+i)Q(M+j)S}}$$
(B.2.4)

Equation B.2.2 is used until Q is 0 or negative. Equation B.2.3 is used until S is 0 or -1. Separate recursion relations are derived for reducing the value of (M+P) depending on whether S and Q are zero.

Integrating the inner integral in Equation B.2.1 by parts

$$z_{\text{B2NPQMS}} = \frac{1}{P+1} x^{P+1} \int_{0}^{1} y^{M} (R_{0}(y))^{\frac{1}{N}} (\Lambda_{x}(y))^{\frac{1}{N}} (R_{x}(y))^{\frac{1}{N}} dy \bigg]_{x=0}^{1}$$

$$-\frac{1}{P+1} \sum_{i=0}^{2} \sum_{j=0}^{2-i} \sum_{j=0}^{1} i N \Lambda_{ij} Z_{B2(N-2)(P+i)Q(M+j)} S$$

$$-\frac{1}{P+1} \sum_{i=0}^{2} \sum_{j=0}^{2-i} \sum_{j=0}^{1} i Q R_{ij} Z_{B2N(P+i)(Q-2)(M+j)} S$$
(B.2.5)

and the same of th

Applying equation B.2.2 to the second term and B.2.4 to the last term in equation B.2.5 and then subtracting the combination of equations B.2.2 and B.2.4 from it

$$0 = x^{P+1} \qquad {}^{Z}_{A5NQMS} \quad ([\Lambda_x], [R_x], [R_0])_{x=0}^{1}$$

$$x Z_{B2(N-2)(P+i+k)(Q-2)(M+j+l)S}$$

The sum of the subscripts in the P and M positions is $(i+j+k+\ell+P+M)$. This is highest when i+j=2; $k+\ell=2$. Separating all terms that meet this criterion,

$$= x^{P+1} \quad {}^{Z}_{A5NQMS} \quad (\dots) \quad {}^{1}_{x=0}$$

$$x^{2}B2(N-2)(P+i+k)(Q-2)(M+j+l)S$$

or, replacing N by N+2, P by P-4, Q by Q+2, throughout,

The recursion implications of equation B.2.6 are shown in Figure B.2.1. In the P-M grid as shown in Figure B.2.2, it is assumed that all $Z_{B2...}$ for which $P+M \le n-1$ are known. Let $Z_{Pn} \stackrel{\triangle}{=} Z_{B2NPQ(n-P)S}$ where all Z_{Pn} are to be determined. The application of equation B.2.6 for all Z_{Pn} (P=3 to n) gives (n-2) equations to be solved simultaneously. If, as is sometimes the case, $Z_{(P-1)n'} \stackrel{Z}{=} (P-2)n' \stackrel{Z}{=} (P-3)n$ and $Z_{(P-4)n}$ are known, equation B.2.6 written in the form

 $\times Z_{B2N(P-(4-i-k))Q(M+j+l)}$ S

is useful for finding $z_{\rm B2NPQMS}$.

(B.2.7)

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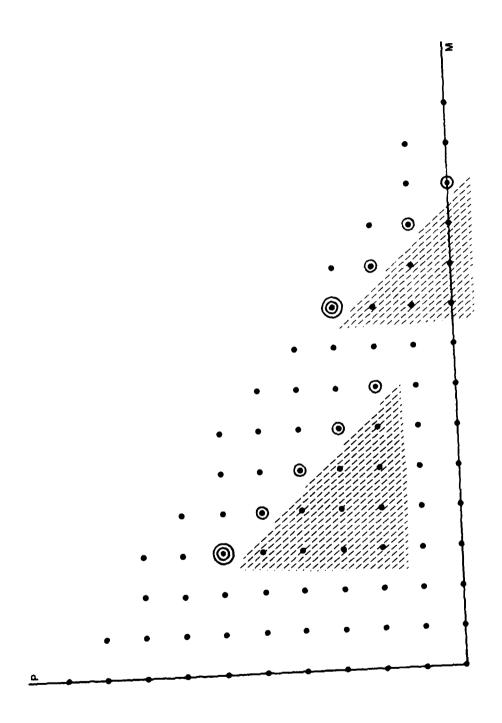


Figure B.2.1

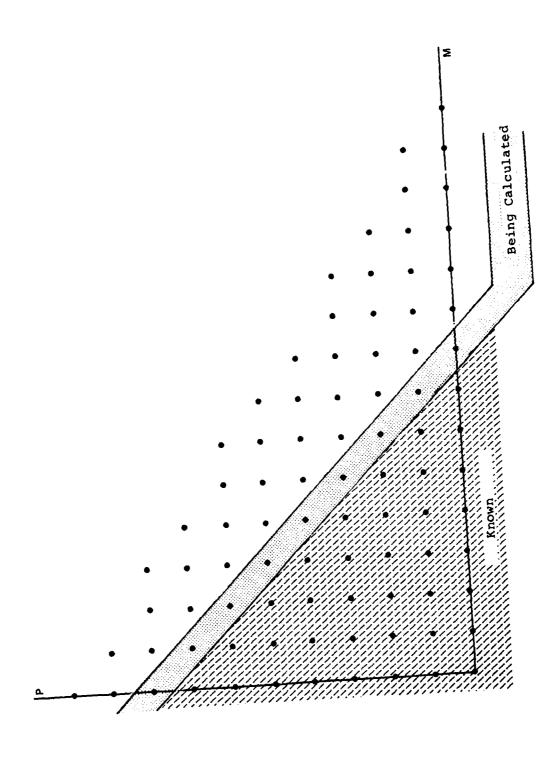


Figure B.2.2

If Q = 0, following a procedure similar to that used in the development of equation B.1.6, an equation simpler than B.2.6 is derived:

$$\sum_{i=0}^{2} \frac{(\frac{1}{2}(N+2)i+P-1)\Lambda_{i}(2-i)}{(2-i)} \sum_{B2N(P-(2-i))0(M+2-i)S}$$

$$= x^{P-1} \sum_{A2(N+2)MS} ([\Lambda_{x}], [R_{0}]) \Big]_{x=0}^{1}$$

$$- \sum_{i=0}^{L} \sum_{j=0}^{D-i} \frac{(\frac{1}{2}(N+2)i+P-1)\Lambda_{ij}}{(B\cdot 2\cdot 8)} \sum_{B2N(P-(2-i))0(M+j)S}$$
(B.2.8)

Other equations are generated depending on whether Q and S are zero by integrating the outer integral in equation B.2.1 by parts

$$z_{\text{B2NPQMS}} = \frac{1}{M+1} \left\{ y^{\text{M+1}} \left(R_0(y) \right)^{\frac{1}{2}S} z_{\text{A2NPQ}} \left(\left[\Lambda_y \right], \left[R_y \right] \right) \right\}_{y=0}^{1}$$

$$- \sum_{i=0}^{2} \sum_{j=0}^{2-i} \frac{1}{2}N^{j} \Lambda_{ij} z_{\text{B2}(N-2)}(p+i)Q(M+j)S$$

$$- \sum_{i=0}^{2} \sum_{j=0}^{2-i} \frac{1}{2}Q^{j} R_{ij} z_{\text{B2NPQ}(M+j)}(Q-2)(M+j)S$$

$$- \sum_{j=0}^{2} \frac{1}{2} S^{j} R_{0j} z_{\text{B2NPQ}(M+j)}(S-2)$$

$$(B.2.9)$$

If both Q and S are zero, equation B.2.9 reduces to

$$z_{\text{B2NP0M0}} = \frac{1}{M+1} y^{M+1} \qquad z_{\text{A2NP0}} (...) \int_{y=0}^{1} \frac{1}{y^{2}} dy^{M+1}$$

$$- \frac{1}{M+1} \sum_{i=0}^{2} \sum_{j=0}^{2-i} \frac{1}{2^{i}} N_{j} \Lambda_{ij} z_{\text{B2(N-2)(P+i)0(M+j)0}}$$

Comparing with equation B.2.4,

$$0 = y^{M+1}$$
 $z_{A2NP0}(...)$ $y=0$

or separating the terms for which i + j = 2 while replacing N by N + 2, M by M - 2 throughout

$$\sum_{j=0}^{2} (M-1+\frac{1}{2}(N+2)j) \wedge (2-j)j^{2}B2N(P+2-j)0(M-(2-j))0$$

$$= y^{M-1} \qquad Z_{A2(N+2)P0} (...) \Big]_{y=0}^{1}$$

$$- \sum_{j=0}^{2} \sum_{i=0}^{\infty} (M-1+\frac{1}{2}(N+2)j) \wedge_{ij} Z_{B2N(P+i)0(M-(2-j))0}$$
(B.2.10)

Figure B.2.3 shows the recursion formula pictorially.

For S=0, $Q\neq 0$, equation B.2.9 becomes, by application of equation B.2.2 to the second term and equation B.2.4 to the third term,

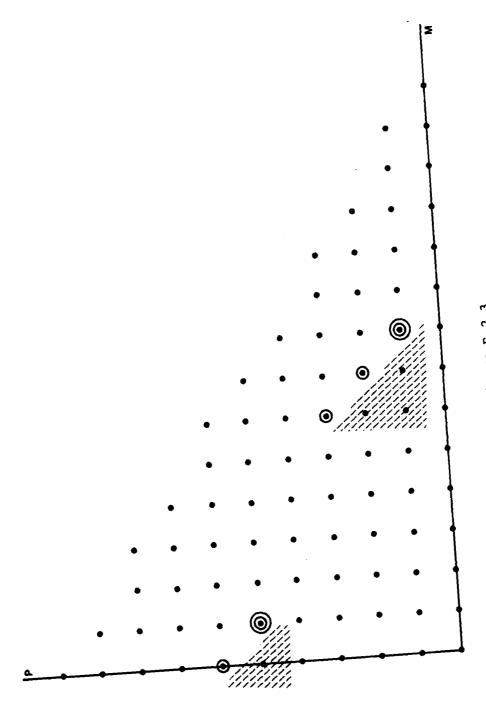


Figure B.2.3

12.0

$$z_{\text{B2NPQM 0}} = \frac{1}{M+1} y^{M+1} \qquad z_{\text{A2NPQ}} \left[\left[\Lambda_{y} \right], \left[R_{y} \right] \right]_{y=0}^{1}$$

$$- \frac{1}{M+1} \sum_{i=0}^{2} \sum_{j=0}^{2-i} \sum_{k=0}^{2-k} \sum_{k=0}^{(\frac{1}{2}N_{j} + \frac{1}{2}Qk)} \Lambda_{ij} R_{kk}$$

$$^{Z}B2(N-2)(P+i+k)(Q-2)(M+j+l)0$$
 (B.2.11)

Combining equations B.2.2 and B.2.4 and subtracting the result from the above,

$$\begin{array}{lll}
0 & = & y^{M+1} & z_{A2NPQ} & (\dots) & \int_{y=0}^{1} \\
& - & \sum_{i=0}^{2} & \sum_{j=0}^{2} & \sum_{k=0}^{2} & (\frac{1}{2}Nj + \frac{1}{2}Qk + M + 1) & \Lambda_{ij} & R_{kk}
\end{array}$$

$$^{\rm Z}$$
B2(N-2) (P+i+k) (Q-2) (M+j+1) 0

or

$$0 = y^{M+1} \qquad Z_{A2NPQ}(...) \int_{y=0}^{1} \\ -\sum_{\substack{\Sigma \\ j=0 \ i=0}}^{\infty} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} (\frac{1}{2}N_j + \frac{1}{2}Q_k + M + 1) \Lambda_{ij} R_{kk}$$

$$Z_{B2(N-2)(P+i+k)(Q-2)(M+j+k)0} \qquad (B.2.12)$$

Separating all terms for which i + j + k + l = 4,

x
 z $_{B2(N-2)}$ (P+i+k) (Q-2) (M+j+ ℓ) 0

or replacing N by N+2, Q by Q+2, M by M-4 throughout,

. .

$$\sum_{j=0}^{2} \sum_{\ell=0}^{2} \left[\frac{1}{2} (N + 2)j + \frac{1}{2} (Q + 2)\ell + M - 3 \right] \Lambda_{(2-j)j} R_{(2-\ell)\ell} Z_{B2N(P+4-j-\ell)Q(M-(4-j-\ell))0}$$

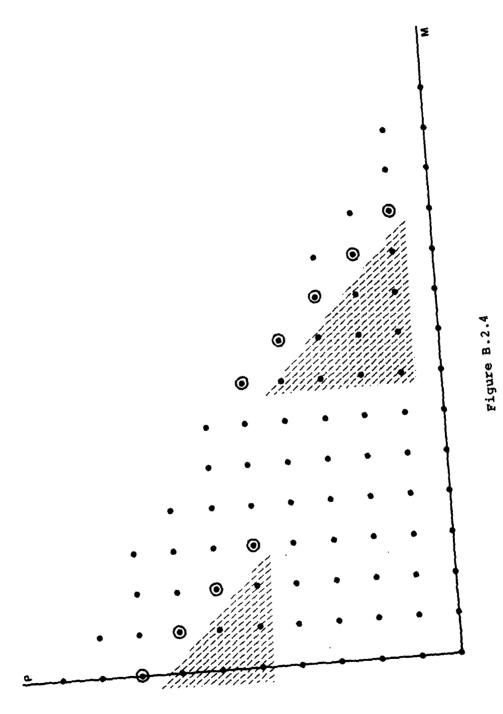
$$= y^{M-3} \qquad z_{A2(N+2)P(Q+2)} (...) \bigg]_{y=0}^{1}$$

$$^{\times}$$
 Z B2N(P+i+k)Q(M-(4-j-l))0 (B.2.13)

Figure B.2.4 shows this relation pictorially.

For Q=0, $S\neq 0$, combining equations B.2.3 and B.2.4 and subtracting the result from equation B.2.9 after application of equation B.2.3 to the second term and equation B.2.4 to the last term,

$$0 = y^{M+1} (R_0(y))^{\frac{1}{2}S} Z_{Al NP} (.) \int_{y=0}^{1} Z_{Al NP}$$



Separating all terms for which i + j + m = 4,

$$\sum_{j=0}^{2} (\frac{1}{2} N j + M + 1 + S) \Lambda_{(2-j)j} R_{02} Z_{B2(N-2)(P+2-j)0(M+2+j)(S-2)}$$

$$= y^{M+1} (R_{0}(y))^{\frac{1}{2}S} Z_{A1NP} (.) \Big]_{y=0}^{1}$$

$$^{\times}$$
 Z B2(N-2)(P+i)0(M+j+m)(S-2)

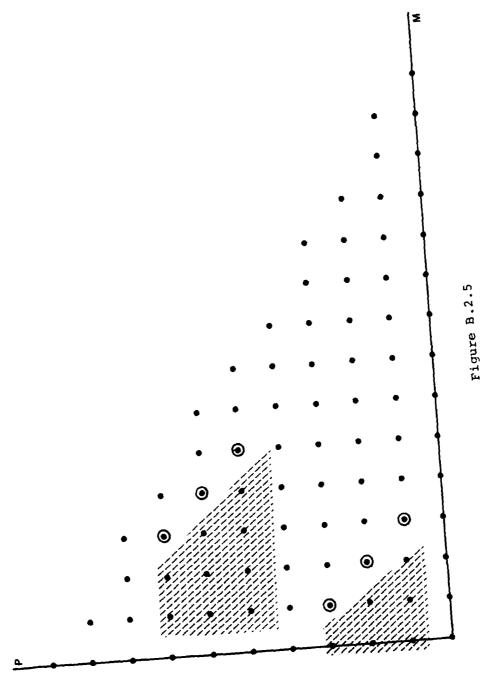
or replacing N by N+2, M by M-4, S by S+2 throughout,

$$\begin{array}{l} 2 \\ \sum\limits_{j=0}^{\Sigma} \left(\frac{1}{2} (N+2) \ j+M-1+S \right) \ \Lambda_{(2-j) \ j} \ ^{R}_{02} \ ^{Z}_{B2N(P+2-j)0 \ (M-(2-j)) S} \\ \\ = y^{M-3} \left(R_{0}(y) \right)^{\frac{1}{2} (S+2)} \ Z_{Al \ (N+2) \ P} \ (.) \end{array} \right]_{y=0}^{1} \\ \\ \begin{array}{l} 2 \\ -\Sigma \ \Sigma \ \Sigma \ \Sigma \ j=0 \ i=0 \end{array} \quad \begin{array}{l} \sum\limits_{m=0}^{\infty} \left(\frac{1}{2} (N+2) \ j+M-3+\frac{1}{2} \left(S+2 \right) \ m \right) \Lambda_{ij} \ ^{R}_{0m} \\ \end{array}$$

$$x Z_{B2N(P+i)0(M-(4-j-m))S}$$
 (B.2.15)

Figure B.2.5 shows this relation pictorially.

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If both Q and S are non-zero, subtracting the combination of equations B.2.2, B.2.3 and B.2.4 from equation B.2.9 after applying equations B.2.2 and B.2.3 on the second, equations B.2.3 and B.2.4 on the third and equations B.2.4 and B.2.2 on the last terms,

$$0 = y^{M+1} (R_0(y))^{\frac{1}{2}S} z_{A2NPQ}([\Lambda_y], [R_y]) \Big]_{y=0}^{1}$$

$$= \sum_{\substack{j=0 \ i=0}}^{2} \sum_{\ell=0}^{2-j} \sum_{k=0}^{2} \sum_{m=0}^{2} \left[\frac{1}{2}N_j + \frac{1}{2}Q\ell + M+1 + \frac{1}{2}Sm \right] \Lambda_{ij} R_{k\ell} R_{0m}$$

$$x = Z_{B2(N-2)}(P+i+k)(Q-2)(M+j+\ell+m)(S-2)$$
 (B.2.16)

Replacing N by N+2, Q by Q+2, M by M-6, S by S+2 throughout while separating the terms for which $i+j+k+\ell+m=6$, equation B.2.16 becomes,

$$\sum_{j=0}^{2} \sum_{\ell=0}^{2} \left[\frac{1}{2} (N+2) j + \frac{1}{2} (Q+2) \ell + M - 3 + S \right] \Lambda_{(2-j) j} R_{(2-\ell) \ell} R_{02}$$

$$\times Z_{B2N}(P+4-i-j-\ell) Q (M-(4-j-\ell)) S$$

$$= y^{M-5} \left(R_{0}(y) \right) Z_{A2}(N+2) P (Q+2) \dots \int_{y=0}^{1} \frac{1}{y=0}$$

$$\sum_{j=0}^{2} \sum_{i=0}^{2} \sum_{\ell=0}^{2} \sum_{m=0}^{2} \sum_{m=0}^{2} \left[\frac{1}{2} (N+2) j + \frac{1}{2} (Q+2) \ell + M - 5 + \frac{1}{2} (S+2) m \right]$$

$$x = \int_{ij}^{R} R_{k\ell} = R_{0m} = Z_{B2N(P+i+k)Q(M-(6-j-\ell-m))S}$$
 (B.2.17)

Figure B.2.6 shows this relation.

The case where Q = S = 0 occurs whenever the far-field approximation is being considered. Calculation strategies are developed here for the Q = S = 0 and the general case $Q \neq 0$, $S \neq 0$. The other two cases Q = 0, $S \neq 0$ and $Q \neq 0$, S = 0 occur infrequently enough to justify not programming them as special cases but rather to use the general, even when they occur. Figure B.2.7 summarizes the calculation strategy for calculating Z_{B2NPOMS} . The relation

$$z_{B2NPQMS} = \frac{1}{2} \sum_{i=1}^{6} w_i (y_i)^M (R_0(y_i))^{S/2} z_{A2NPQ} ([\Lambda_{y=y_i}], [R_{y=y_i}])$$

(B.2.18)

is used for numerical integration. Here, $y_i = \frac{1}{2}(x_i + 1)$ and w_i and x_i are given by Abramowitz and Stegun (Table 25.4, n=6).

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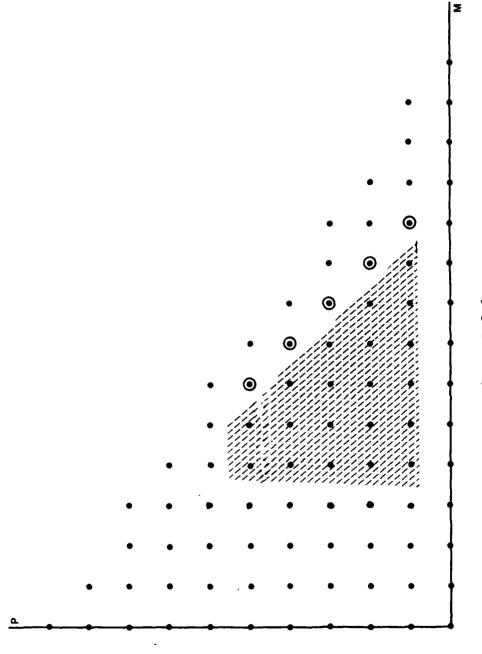


Figure B.2.6

Case 1 of 2 on first page

Case 2 of 2 on second page

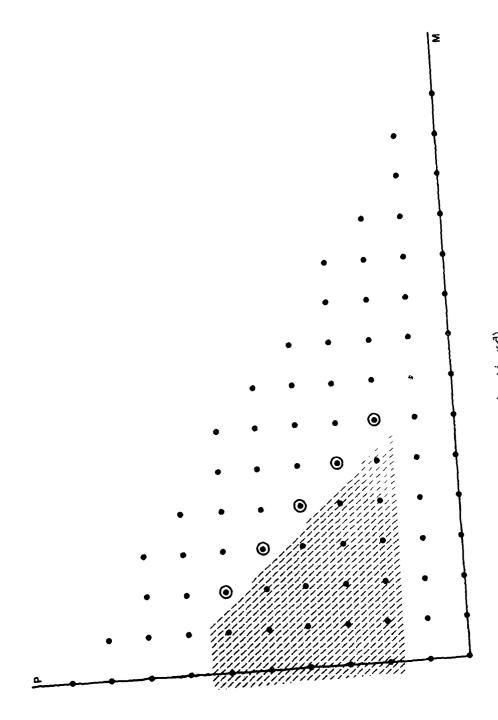
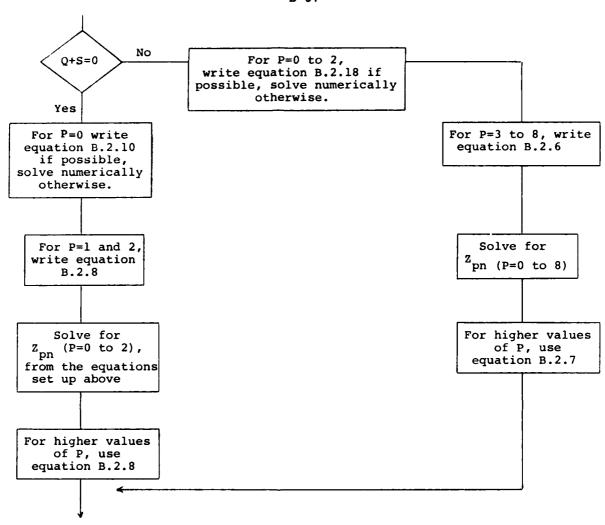


Figure B.2.6 (continata)



Calculation of Z_{B2NPQMS} for various values of P and M (and given values of N, Q and S)

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APPENDIX C

CLOSED FORM SOLUTION FOR SOME TRIPLE INTEGRALS ARISING IN
THE COMPUTATION OF MUTUAL IMPEDANCE QUANTITIES

Notation:

$$R = R(x,y,z) = R_{x}(y,z) = R_{y}(z,x) = R_{z}(x,y) = R_{xy}(z)$$

$$= R_{yz}(x) = R_{zx}(y) = \sum_{\substack{i=0 \ i=0}}^{2} \sum_{\substack{i=0 \ k=0}}^{2-i-j} R_{ijk} x^{i} y^{j} z^{k}.$$

Writing $R_{_{\mathbf{x}}}(\mathbf{y},\mathbf{z})$ implies the form

$$R_{\mathbf{x}}(\mathbf{y},\mathbf{z}) = \sum_{\mathbf{j}=0}^{2} \sum_{\mathbf{k}=0}^{2-\mathbf{j}} \begin{bmatrix} 2-\mathbf{j}-\mathbf{k} \\ \Sigma \\ \mathbf{i}=0 \end{bmatrix} \mathbf{y}^{\mathbf{j}} \mathbf{z}^{\mathbf{k}},$$

i.e., the variable appearing in the subscript is the most suppressed. This definition is consistent with the statement that $\begin{bmatrix} R_{\mathbf{x}} \end{bmatrix}$ is the matrix

matrix
$$\begin{bmatrix} R_{x} \end{bmatrix} = \begin{bmatrix} R_{000} + R_{100} \times + R_{200} x^{2} & R_{001} + R_{101} x & R_{002} \\ R_{010} + R_{110} \times & R_{011} & 0 \\ R_{020} & 0 & 0 \end{bmatrix}$$

Similar notation applies to the subscripts y and z. Consistent with this notation,

$$R_0(y,z) = \sum_{\substack{j=0 \\ j=0}}^{2} \sum_{k=0}^{2-j} R_{0jk} y^j z^k$$

and $\begin{bmatrix} \Re_{\mathbf{x}=\mathbf{0}} \end{bmatrix}$ represents the matrix

$$\begin{bmatrix} R_{000} & R_{001} & R_{002} \\ R_{010} & R_{011} & 0 \\ R_{020} & 0 & 0 \end{bmatrix}$$

Writing R as R_{yz} (x) implies the form

$$R_{yz}(x) = \sum_{i=0}^{2} \begin{bmatrix} 2-i & 2-i-j \\ \sum & \sum \\ j=0 & k=0 \end{bmatrix} x^{i}$$

and is consistent with the statement that $\begin{bmatrix} R \\ yz \end{bmatrix}$ is the vector

$$\begin{bmatrix} 2 & 2-j & & & & \\ \sum_{j=0}^{\Sigma} & \sum_{k=0}^{\Sigma} & R_{0jk} & y^{j}z^{k} & & \\ & 1 & 1-j & & & \\ \sum_{j=0}^{\Sigma} & \sum_{k=0}^{\Sigma} & R_{1jk} & y^{j}z^{k} & \\ & & & R_{200} & & \end{bmatrix}$$

Consistent with the above definitions, $\begin{bmatrix} R & (y=0) & (z=0) \end{bmatrix}$ implies the vector

C.1 The Computation Of $^{\rm Z}$ _{ClPQMSGH}

By definition,

$$Z_{\text{C1PQMSGH}} = \int_{0}^{1} z^{G} (R_{00}(z))^{H/2} \int_{0}^{1} y^{M} (R_{0}(y,z))^{S/2}$$

$$\int_{0}^{1} x^{P} (R(x,y,z))^{Q/2} dxdydz \qquad (C.1.1)$$

Writing $(.)^n = (.)^{n-2} (.)^2$ it is easily proved that

and

$$z_{\text{ClPQMSGH}} = \sum_{k=0}^{2} R_{00k} z_{\text{ClPQMS}(G+k)(H-2)}$$
(C.1.4)

Equations C.1.2 through C.1.4 are used as recursion relations for desired values of Q, S and H in terms of (smaller) given values of Q, S and H. Relations for giving \mathbf{Z}_{Cl} for desired values of P, M and G in terms of given (smaller) values of P, M and G are derived as follows:

Integrating the integral on x by parts

$$Z_{\text{C1PQMSGH}} = \frac{1}{P+1} x^{P+1} \int_{0}^{1} z^{G} (R_{00}(z))^{P/2} \int_{0}^{1} y^{M} (R_{0}(y,z))^{S/2} (R_{x}(y,z))^{Q/2} dydz \bigg]_{x=0}^{1}$$

(C.1.5)

Comparing with equation C.1.2,

$$0 = x^{P+1} Z_{B2QMSGH} ([R_x], [R_{x=0}]) \begin{cases} 1 \\ x=0 \end{cases}$$

$$- \sum_{i=0}^{2} \sum_{j=0}^{2-i} \sum_{k=0}^{2-i-j} (\frac{1}{2} iQ + P+1) R_{ijk} Z_{C1(P+i)(Q-2)(M+j)S(G+k)H}$$

(C.1.6)

Separating the terms for which i+j+k=2, which changing P to P-2, Q to Q+2

... with some

x
 z C1(P+i-2)Q(M+j)S(G+k)H (C.1.7)

The implications of this recurrence relation are shown in Figure C.1.1. For a given value of Q,S and H, different $^{\rm Z}_{\rm C1PQMSGH}$ values map onto a grid in the P-M-G coordinate system. The points of the grid with two circles around them (A and L) are the given values of $^{\rm P}_{\rm 0}$, $^{\rm M}_{\rm 0}$ and $^{\rm G}_{\rm 0}$. When $^{\rm P}_{\rm 0}$, (Point A) equation C.1.7 gives a linear combination of points A through F in terms of points G, H, J and K. When $^{\rm P}_{\rm 0}$ =1 (Point L), a linear combination of points L, M and N is given in terms of point P. A plane of constant P+M+G is shown in Figure C.1.2 Recursion relation C.1.7 can be applied to all points for which P>1. n+1 more equations must be generated (for P=0) in order that all points in this plane may be solved. This is done by integrating the integral on y by parts to give

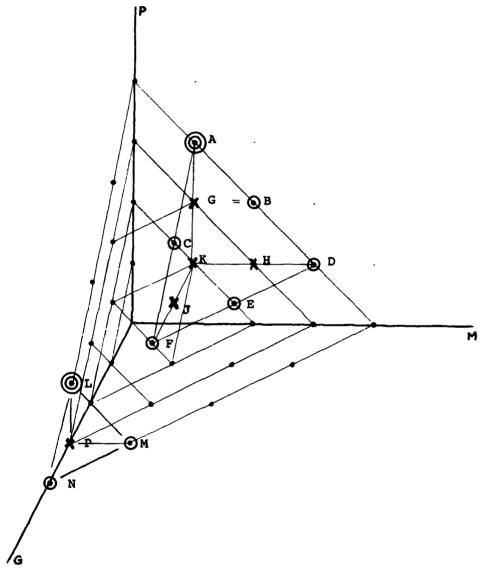


Figure C.1.1: Implications of the Recursion Relation C.1.7

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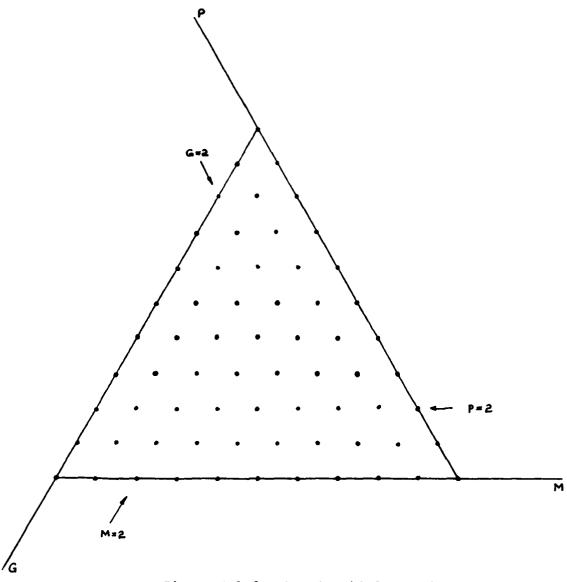


Figure C.1.2: P-M-G Grid for a Given Value of P+M+G

Substituting equation C.1.3 into the second term and equation C.1.2 into the third term, comparing with a combination of equations C.1.2 and C.1.3, replacing Q by Q+2, M by M-4 and S by S+2 throughout and writing

$$z_{B3PQHGS}$$
 ([A], [B]) = $\int_{0}^{1} (A(y))^{H/2} y^{G} (B_{0}(y))^{S/2}$
 $\times \int_{0}^{1} x^{P} (B(x,y))^{Q/2} dxdy$ (C.1.9)

where [B] is a matrix whose first row is B_0 and [A] is a vector, the following equation is obtained

$$0 = y^{M-3} Z_{B3P(Q+2)HG(S+2)} ([R_{0C}], [R_{y}]) \begin{bmatrix} 1 \\ y=0 \end{bmatrix}$$

$$= 2 \qquad 2-i \qquad 2-i-j \qquad 2 \qquad 2-\ell$$

$$- \sum_{i=0}^{\Sigma} \sum_{j=0}^{\Sigma} \sum_{k=0}^{\Sigma} \sum_{m=0}^{\Sigma} (\frac{1}{2} (Q+2)j + M-3 + \frac{1}{2} (S+2))$$

 $x = R_{ijk}R_{0 \ell m} = C_{1(P+i)Q(M+j+\ell-4)S(G+k+m)H}$

* Techniques for solving $Z_{\text{B3PQHGS}}([A],[B])$ are similar to those presented in Appendix B.2 for $Z_{\text{B2NPQMS}}([\Lambda],[R])$.

or, separating the terms for which i+j+k=2, $\ell+m=2$,

It is to be noted that equation C.1.7 helps to write z_{C1PQMSGH} in terms of other z_{C1} with smaller values of P but larger values of M and G. Equation C.1.10 gives z_{C1} in terms of other z_{C1} with smaller values of M but larger values of P and G. Another recursion relation that gives z_{C1} in terms of other z_{C1} with smaller values of G but larger values of P and M is derived by integrating the integral on z in equation C.1.1 by parts to give

$$\begin{split} & Z_{\text{C1PQMSGH}} = \frac{1}{G+1} \left\{ z^{G+1} \left(R_{00}(z) \right)^{H/2} \quad Z_{\text{B1PQMS}} \left(\left[R_{z} \right] \right) \right\}_{z=0}^{1} \\ & - \sum_{i=0}^{Z} \quad \sum_{j=0}^{\Sigma} \quad \sum_{k=0}^{\Sigma} \quad \frac{1}{2} Q^{k} R_{ijk} \quad Z_{\text{C1}(P+i)} \left(Q-2 \right) \left(M+j \right) S \left(G+k \right) H \\ & - \sum_{j=0}^{Z} \quad \sum_{k=0}^{Z} \quad \frac{1}{2} \quad S^{k} R_{0jk} \quad Z_{\text{C1PQ}(M+j)} \left(S-2 \right) \left(G+k \right) H \\ & - \sum_{k=0}^{Z} \quad \frac{1}{2} \quad H^{k} R_{00k} \quad Z_{\text{C1PQMS}} \left(G+k \right) \left(H-2 \right) \right\} \end{split}$$

Substituting equations C.1.3 and C.1.4 into the second term, equations C.1.4 and C.1.2 into the third term, equations C.1.2 and C.1.3 into the fourth term, comparing the result with a combination of equations C.1.2, C.1.3 and C.1.4, replacing Q by Q+2, S by S+2, G by G-6, H by H+2 throughout, the following equation is obtained:

$$0 = z^{G-5} (R_{00}(z)) (H+2)/2 z_{B1P(Q+2)M(S+2)} ([R_z]) \begin{bmatrix} 1 \\ z \end{bmatrix} z = 0$$

$$- \sum_{i=0}^{\Sigma} \sum_{j=0}^{\Sigma} \sum_{k=0}^{\Sigma} \sum_{k=0}^{\Sigma} \sum_{m=0}^{\Sigma} \sum_{n=0}^{\Sigma} z$$

$$(\frac{1}{2}(Q+2)k+\frac{1}{2}(S+2)m+G-5+\frac{1}{2}(H+2)n)$$

 R ijk R 0lm R 00n Z C1(P+i)Q(M+j+l)S(G+k+m+n-6)H (C.1.12)

or, separating the terms for which i+j+k=2, $\ell+m-2$, n=2

 2 Cl(P+i)Q(M+j+l)S(G+k+m+n-6)H

(C.1.13)

As suggested in the discussion on Figure C.1.2, equations C.1.7, C.1.10 and C.1.13 can be used for various values of P, M and G in order to generate enough equations to solve for all points on the constant P+M+G plane simultaneously. Figure C.1.3 shows the left-hand side points of equation C.1.7 (i.e., each application of equation C.1.7 yields a linear combination of a cluster such as is shown in Figure C.1.3). Similarly Figure C.1.4 shows the left-hand side points of equation C.1.10 and Figure C.1.5 shows the left-hand side points of equation C.1.13. At least one of the equations is

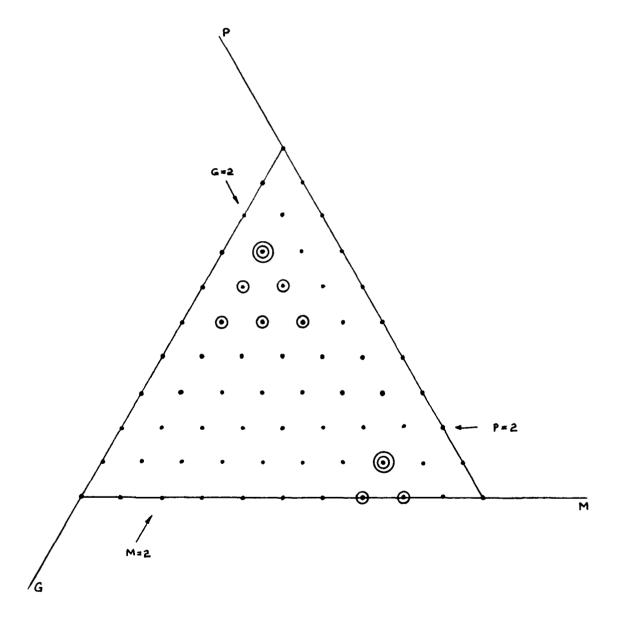
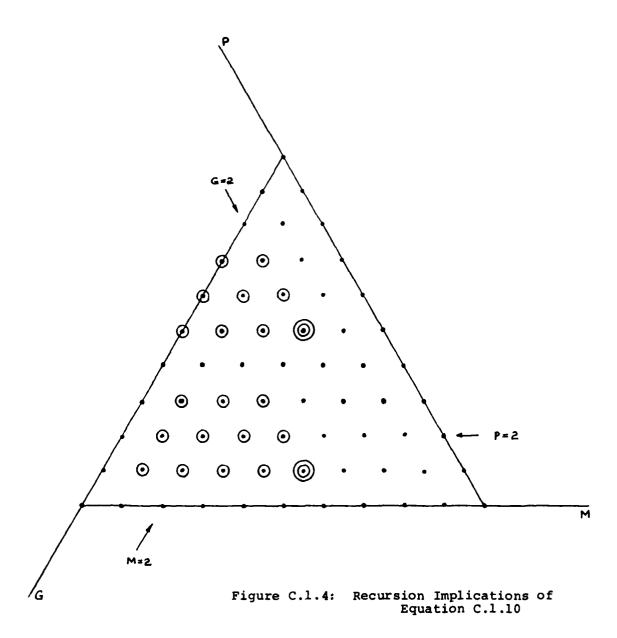


Figure C.1.3: Recursion Implications of Equation C.1.7



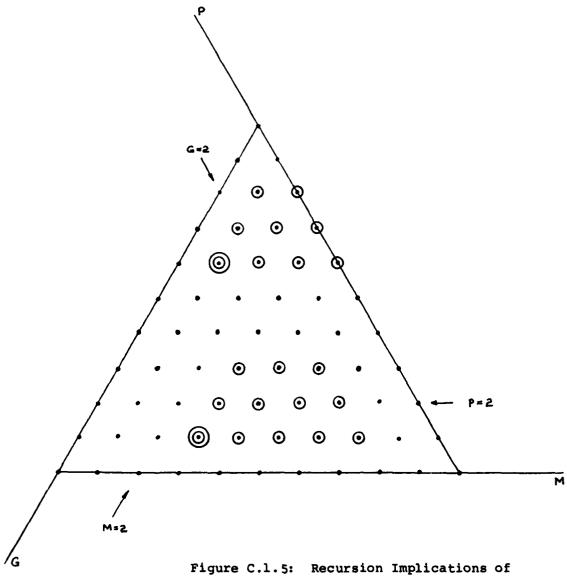


Figure C.1.5: Recursion Implications of Equation C.1.13

applicable at each of the points and is solved simultaneously to give the value of \mathbf{z}_{Cl} at each of the points.

The case where Q=S=H=0 is of special importance because it is involved for the far field case (see Appendix D for details). The three integrals in the definition of \mathbf{z}_{Cl} decouple in this case and

$$z_{C1P0M0G0} = \frac{1}{(P+1)(M+1)(G+1)}$$
 (C.1.14)

C.2 The Computation of $^{\rm Z}_{\rm C2NPQMSGH}$

$$\begin{split} & \text{By definition,} \\ & \text{Z}_{\text{C2NPQMSGH}} = \int_{0}^{1} z^{\text{G}} \left(R_{00}(z) \right)^{\text{H}/2} \int_{0}^{1} y^{\text{M}} \left(R_{0}(y,z) \right)^{\text{S}/2} \int_{0}^{1} \left(\Lambda(x,y,z) \right)^{\text{N}/2} \\ & \text{x}^{\text{P}} \left(R(x,y,z) \right)^{\text{Q}/2} \, \text{dxdydz} \end{aligned} \qquad (C.2.1) \\ & \text{Writing } (.)^{\text{D}} = (.)^{\text{D}-2} \left(. \right)^{2} \, \text{it is easily proved that} \\ & \text{Z}_{\text{C2NPQMSGH}} = \begin{cases} 2 & 2-i & 2-i-j \\ i=0 & j=0 & k=0 \end{cases} & \Lambda_{ijk} \, ^{\text{Z}}_{\text{C2}(N-2)} \left(p+i \right) Q(M+j) \, \text{S} \left(G+k \right) \, \text{H} \\ & \text{C2.2.2} \end{cases} \\ & \text{Z}_{\text{C2NPQMSGH}} = \begin{cases} 2 & \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} R_{ijk} \, ^{\text{Z}}_{\text{C2NPQ}(M+j)} \left(Q-2 \right) \left(M+j \right) \, \text{S} \left(G+k \right) \, \text{H} \\ & \text{C2.3.3} \end{cases} \\ & \text{Z}_{\text{C2NPQMSGH}} = \begin{cases} 2 & \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} R_{0jk} \, ^{\text{Z}}_{\text{C2NPQ}(M+j)} \left(S-2 \right) \left(G+k \right) \, \text{H} \\ & \text{C2.4.4} \end{cases} \\ & \text{Z}_{\text{C2NPQMSGH}} = \begin{cases} 2 & \sum_{k=0}^{\infty} R_{00k} \, ^{\text{Z}}_{\text{C2NPQMS}(G+k)} \left(H-2 \right) \right. \end{aligned} \qquad (C.2.5) \end{aligned}$$

Equations C.2:2 through C.2.4 are used as recursion relations for desired values of N, Q, S and H in terms of (smaller) given values of N, Q, S and H. Relations for giving \mathbf{Z}_{C2} for desired values of P, M and G are derived as follows:

. .

Integrating equation C.2.1 by parts with respect to the dummy variable x and applying equations C.2.2 and C.2.3,

$$\begin{split} & Z_{\text{C2NPQMSGH}} = \frac{1}{P+1} \left\{ \begin{array}{l} x^{P+1} & \int_{0}^{1} z^{G} \left(R_{00}(z)\right)^{H/2} & \int_{0}^{1} y^{M} \left(R_{0}(y,z)\right)^{S/2} \\ & x \left(\Lambda_{x}(y,z)\right)^{N/2} \left(R_{x}(y,z)\right)^{Q/2} dydz \end{array} \right\}_{x=0}^{1} \\ & - \sum_{i=0}^{2} \sum_{j=0}^{2-i} \sum_{k=0}^{2-i-j} \sum_{\ell=0}^{2} \sum_{m=0}^{2-\ell} \sum_{n=0}^{2-\ell-m} \left(\frac{1}{2}iN + \frac{1}{2}\ell_{Q}\right) \Lambda_{ijk} R_{\ell mn} \\ & \times Z_{C2(N-2)(P+i+\ell)(Q-2)(M+j+m)S(G+k+n)H} \right\} \end{split}$$

Comparing with a combination of equations C.2.2 and C.2.3 and changing N to N+2, P to P-4, Q to Q+2 throughout,

$$0 = x^{P-3} Z_{B3(N+2)(Q+2)MSGH} \left(\begin{bmatrix} \Lambda_x \end{bmatrix}, \begin{bmatrix} R_x \end{bmatrix}, \begin{bmatrix} R_x \end{bmatrix}, \begin{bmatrix} R_{x-0} \end{bmatrix} \right) \begin{bmatrix} 1 \\ x=0 \end{bmatrix}$$

$$- \sum_{i=0}^{2} \sum_{j=0}^{2-i} \sum_{k=0}^{2-i-j} \sum_{\ell=0}^{2} \sum_{m=0}^{2-\ell-m} \sum_{n=0}^{\ell-\ell-m} \left(\frac{1}{2}i(N+2) + p-3 + \frac{1}{2}\ell(Q+2) \right)$$

$$\times$$
 Λ_{ijk} R_{lmn} $C_{2N(P-(4-i-l))Q(M+j+m)S(G+k+n)H}$ (C.2.7)

$$z_{B3NQMSGH}$$
 ([A], [B], [C]) $\triangleq \int_{0}^{1} y^{G} (c_{0}(y))^{H/2} \int_{0}^{1} (A(x,y))^{N/2}$

$$(B(x,y))^{Q/2} x^{M} (C(x,y))^{S/2} dxdy.$$
(C.2.8)*

Separating all terms in equation C.2.7 for which i+j+k=2 and $\ell+m+n=2$, the following recursive relation is obtained:

x
 z C2N (P-(4-i- ℓ))Q(M+j+m)S(G+4-i-j- ℓ -m)H

=
$$x^{P-3} z_{B3(N+2)(Q+2)MSGH} (...) \Big]_{x=0}^{1}$$

^{$$\Lambda$$}ijk ^{R} ℓ mn Z C2N(P-(4-i- ℓ))Q(M+j+m)S(G+k+n)H

(C.2.9)

* Techniques for the solution of $Z_{B3NQMSGH}([A],[B],[C])$ are similar to those presented in Appendix B.2 for $Z_{B2NPQMS}([\Lambda],[R])$.

If equation C.2.9 is written with N=N₀, P=P₀, etc., each of the terms on the left-hand side has P+M+G=P₀+M₀+G₀. The terms on the right-hand side have P+M+G=P₀+M₀+G₀-n where n is 1, 2, 3, or 4. This statement implies that none of the points on the right-hand side of equation C.2.9 fall on the $(P_0+M_0+G_0=\text{constant})$ plane. Figure C.2.1 shows a plane of constant P₀+M₀+G₀. Recursion relation C.2.9 may be applied to all points on this plane for which P₀ \geqslant 3. Equations that may be applied for smaller values of P₀ are developed next. Integrating equation C.2.1 by parts with respect to the dummy variable y and applying two of the equations C.2.2 through C.2.4 to each term except the first,

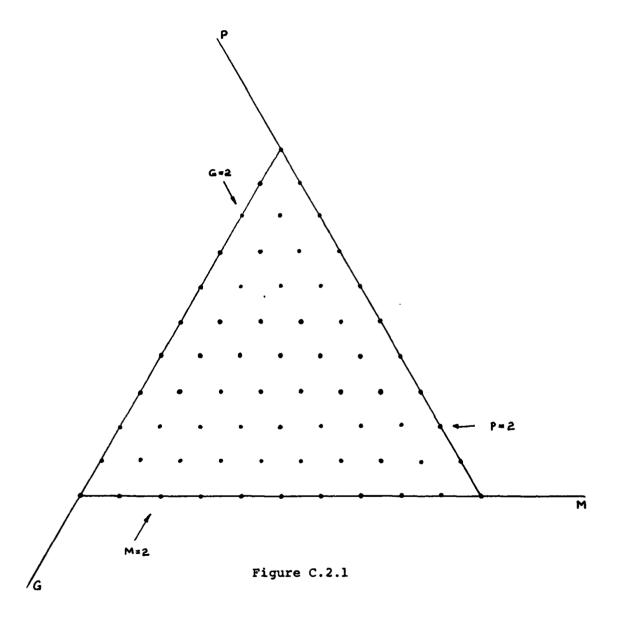
 $^{\rm Z}$ C2NPQMSGH =

$$\frac{1}{M+1} \left\{ y^{M+1} \int_{0}^{1} z^{G} (R_{00}(z))^{\frac{1}{2}} Y^{2} (R_{0}(z))^{S/2} \int_{0}^{1} (\Lambda_{y}(x,z))^{N/2} x^{P} (R_{y}(x,z))^{Q/2} dxdz \right\}_{y=0}^{1}$$

$$- \sum_{i=0}^{2} \sum_{j=0}^{2-i} \sum_{k=0}^{2-i-j} \sum_{k=0}^{2} \sum_{k=0}^{2-k} \sum_{k=0}^{2-k-m} \sum_{k=0}^{2} \sum_{k=0}^{2-p} (\frac{1}{2} ij + \frac{1}{2} Qm + \frac{1}{2} Sp)$$

$$\times \Lambda_{ijk}^{R} R_{lmn}^{R} R_{0pq}^{Z} C_{2}(N-2)(P+i+\ell) (Q-2) (M+j+m+p) (S-2) (G+k+n+q) H \right\}$$

$$(C.2.10)$$



Comparing equation C.2.10 with a combination of equations C.2.2 through C.2.4, changing N to (N+2), Q to (Q+2), M to (M-6), S to (S+2) throughout, and separating all terms for which i+j+k=2, $\ell+m+n=2$, and p+q=2, the following recursion relation is obtained:

$$^{\rm X} \quad ^{\rm \Lambda} (2-j-k) \ jk \ ^{\rm R} (2-m-n) \ mn \ ^{\rm R} 0 p (2-p) \ ^{\rm Z} C2N (P+4-j-k-m-n) \ Q \ (M-(6-j-m-p)) \ S \ (G+k+n+2-p) \ H$$

$$= y^{M-5} z_{B4 \, (N+2) \, P \, (Q+2) HG \, (S+2)} \, ([\Lambda_y], [R_y], [R_{00}]) \Big]_{y=0}^{1}$$

$$(\frac{1}{2} \text{ (N+2) j } + \frac{1}{2} \text{ (Q+2)m} + \text{M-5} + \frac{1}{2} \text{ (S+2)p) } \Lambda_{ijk} R_{lmn} R_{0pq}$$

$$\times Z_{C2N(P+i+l)Q(M-(6-j-m-p))S(G+k+n+q)H}$$
 (C.2.11)

where

$$z_{B4NPQHGS} ([A], [B], [C]) \triangle \int_{0}^{1} (C(y))^{H/2} y^{G} (B_{0}(y))^{S/2}$$

$$\times \int_{0}^{1} (A(x,y))^{N/2} x^{P} (B(x,y))^{Q/2} dx dy \qquad (C.2.12)*$$

^{*} Techniques for the solution of $Z_{BANPQHCS}$ are similar to those presented in Appendix B.2 for $Z_{B2NPOMS}([\Lambda],[R])$.

Here, [A] and [B] are matrices, B_0 is the first row of the matrix [B] and [C] is a vector.

Integrating equation C.2.1 by parts with respect to the dummy variable z and applying three of equations C.2.2 through C.2.5 to each term except the first,

$$\begin{split} & Z_{\text{C2NPQMSGH}} = \frac{1}{G+1} \left\{ z^{G+1} \left(R_{00}(z) \right)^{H/2} \int_{0}^{1} y^{M} \left(R_{0z}(y) \right)^{S/2} \int_{0}^{1} \left(\Lambda(x,y,z) \right)^{N/2} \right. \\ & \left. \times p_{\left(R(x,y,z) \right)} Q^{/2} \right. dx dy \right]_{z=0}^{1} \end{split}$$

$$x = \Lambda_{ijk} R_{lmn} R_{0pq} R_{00s} Z_{C2(N-2)(P+i+l)(Q-2)(M+j+m+p)(S-2)(G+k+n+q+s)(H-2)}$$
(C.2.13)

Comparing equation C.2.13 with a combination of equations C.2.2 through C.2.5, changing N to (N+2), Q to (Q+2), S to (S+2), G to (G-8) and H to (H+2) throughout and separating all terms for which i+j+k=2, $\ell+m+n=2$, p+q=2 and s=2, the following recursion relation is obtained:

 $^{\text{x } \Lambda_{\text{i}}(2-k-i)k} \, ^{\text{R}} \ell_{(2-n-\ell)n} \, ^{\text{R}} 0 \, (2-q)q \, ^{\text{R}} 0 0 2 \, ^{\text{Z}} \text{C2N}(P+i+\ell)Q \, (\text{M+6-k-i-n-}\ell-q) \, \text{S} \, (\text{G-}(6-k-n-q)) \, d^{\text{R}} \, d^{$

$$= z^{G-7} (R_{00}(z))^{(H+2)/2} z_{B2(N+2)P(Q+2)M(S+2)} ([\Lambda_z], [R_z]) \Big]_{z=0}^{1}$$

$$(\frac{1}{2} \text{ (N+2)k} + \frac{1}{2} \text{ (Q+2)n} + \frac{1}{2} \text{ (S+2)q +G-7} + \frac{1}{2} \text{ (H+2)s)} \Lambda_{ijk}^{R} \text{lmn}^{R} \text{Opq}^{R} \text{O0s}$$

 $\times Z_{C2N(P+i+l)Q(M+j+m+p)S(G-(8-k-n-q-s))H}$

(C.2.14)

One of equations C.2.9, C.2.11 or C.2.14 is applied for each (P, M, G) in Figure C.2.1. The resulting equations are solved simultaneously to give each of the \mathbf{Z}_{C2} A much simpler set of equations is generated for $\mathbf{Q}=\mathbf{S}=\mathbf{H}=\mathbf{0}$. This condition applies in the far field case (see Appendix D). The first of the farfield case equations is obtained by integrating the equation

$$z_{\text{C2NP0M0G0}} = \int_{0}^{1} z^{G} \int_{0}^{1} y^{M} \int_{0}^{1} (\Lambda(x,y,z))^{N/2} x^{P} dxdydz$$
(C.2.15)

by parts with respect to the dummy variable x to give

$$z_{\text{C2NP0M0G0}} = \frac{1}{P+1} \left\{ x^{P+1} \int_{0}^{1} z^{G} \int_{0}^{1} y^{M} \left(\Lambda_{x}(y,z) \right)^{N/2} dydz \right\}_{x=0}^{1}$$

$$- \sum_{i=0}^{2} \sum_{j=0}^{2-i} \sum_{k=0}^{2-i-j} \frac{1}{2} \text{Ni} \Lambda_{ijk} z_{\text{C2(N-2) (P+i) 0 (M+j) 0 (G+k) 0}}_{(C.2.16)}$$

Comparing with equation C.2.2, replacing N by N+2 and P by P-2 and separating all terms for which i+j+k=2, the recursive relation

$$\sum_{i=0}^{2} \sum_{j=0}^{2-i} (\frac{1}{2}(N+2)i+P-1) \Lambda_{ij} (2-i-j)^{2} C2N(P-(2-i)) 0 (M+j) 0 (G+2-i-j) 0$$

$$= x^{P-1} Z_{BlM(N+2)G0} ([\Lambda_{x}])^{1} X_{x=0}$$

$$- \sum_{i=0}^{2} \sum_{j=0}^{2} \sum_{k=0}^{\infty} (\frac{1}{2}(N+2)i+P-1) \Lambda_{ijk}^{2} Z_{C2N(P-2+i)} 0 (M+j) 0 (G+k) 0$$

$$(C.2.17)$$

is obtained. This relation can be applied for all P>1. A second far field case equation is obtained by integrating equation C.2.15 by parts with respect to the dummy variable y to give

' 1

$$z_{\text{C2NP0M0G0}} = \frac{1}{M+1} \left\{ y^{M+1} \int_{0}^{1} z^{G} \int_{0}^{1} (\Lambda_{y}(x,z))^{N/2} x^{P} dxdz \right\}_{y=0}^{1}$$

Comparing with equation C.2.2, replacing N by N+2 and M by M-2 and separating all terms for which i+j+k=2, the recursive relation

This relation can be applied for all M≯1. A third for field equation

$$\sum_{k=0}^{2} \sum_{i=0}^{2-k} \frac{1}{2} (N+2)k+G-1) \Lambda_{i} (2-k-i)k^{-2} C2N(P+i) 0 (M+2-k-i) 0 (G-(2-k)) 0$$

$$= \mathbf{z}^{G-1} Z_{B1P(N+2)M0} ([\Lambda_{\mathbf{z}}]) \Big]_{z=0}^{1}$$

$$- \sum_{i=0}^{2} \sum_{j=0}^{2-k} \sum_{k=0}^{2-k} \frac{1-i-j-j}{k} \sum_{k=0}^{2-k-i} \frac{1}{2} (N+2)k+G-1) \Lambda_{ijk} Z_{C2N(P+i) 0 (M+j) 0 (G-(2-k)) 0}$$

$$(C.2.20)$$

is derived similarly. This relation can be used for all G \geqslant 1. A constant P+M+G plane is shown in figure C.2.2 and a strategy for calculating $^2\text{C2NP0M0G0}$ is outlined.

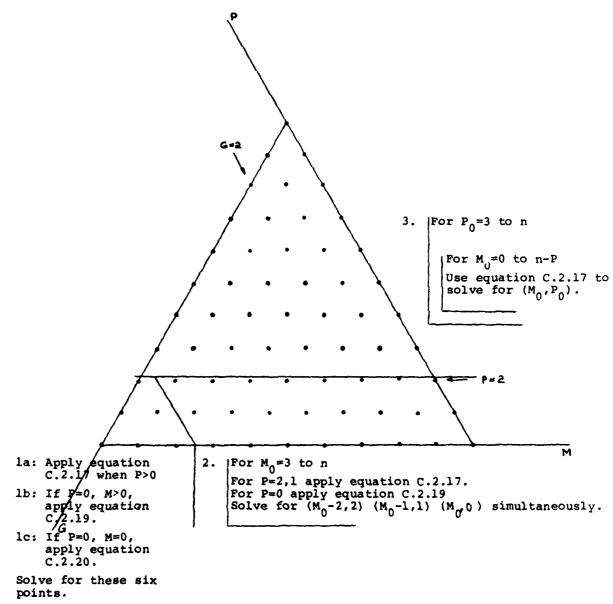


Figure C.2.2

APPENDIX D

THE NUMERICAL PROCEDURE FOR THE COMPUTATION OF $z_0^{WW},\ z_0^{WS},\ z_0^{SW},\ \text{and}\ z_0^{SS}$

Equations for the computation of z_0^{WW} , z_0^{WS} , z_0^{SW} and z_0^{SS} have been developed in sections 2.3, 2.4 and 2.5. These are developed further in this appendix to a form that is more convenient for computational purposes. The partial sums are subject to a high degree of instability when the distances are large compared to the sizes of the segments. For this case, separate far field* solutions have been developed.

Equations 2.3.8 and 2.3.12 are used for computing the mutual impedances from z_0^{WW} . For the purpose of computing z_0^{WW} , it is appropriate to collect terms of order (n+p) together. Writing q = n+p, and noting that

^{*}Traditionally, far field implies a region far enough so that the field on the receptor can be considered a constant. This is not the implication here when the term far field is used. Instead, far field implies a region far enough so that ℓ/r , the ratio of segment size to distance between the receiver and transmitter origins is small enough that $(\ell/r)^3$ is negligible compared to 1.

where m' = m - n, equation 2.3.18 can be written as

$$z_0^{WW} = \sum_{\sigma=1}^{2} \frac{(-1)^{\sigma s} \left(\exp -jkr_{110}\right) \ell_{\sigma}^{s+1} \left(\Lambda_{0}^{\zeta}\right)^{\frac{1}{2}\nu} r_{110}^{\alpha}}{2j \sin k\ell_{\sigma}}$$

$$x \stackrel{5}{\underset{i=0}{\Sigma}} a_{i} \left(jk \ell_{\sigma} \right)^{i} \stackrel{i}{\underset{q=0}{\Sigma}} \left(1 - (-1)^{i+q} \right) \left(\frac{r_{110}}{\ell_{\sigma}} \right)^{q} \stackrel{i-q}{\underset{m'=0}{\Sigma}} \left(i \atop m'+q \right) \left(m'+q \right) \left(-1 \right)^{m'}$$

$$\begin{array}{ccc}
x & \Sigma \\
n=0 & {q \choose n} & (-1)^n & Z_{A2\nu(s+m')(n+\alpha)} & ([\Lambda],[R])
\end{array} (D.1)$$

When $\ell_{\sigma} << r_{110}$, all $z_{A2\nu(s+m')(n+\alpha)}$ are of the order of $z_{A2\nu(s+m')0}$ and the innermost summation results in the subtraction of almost equal quantities. Therefore, it is appropriate and convenient to express the quantity

$$\sum_{n=0}^{q} {q \choose n} (-1)^n Z_{A2\nu(s+m')(n+\alpha)} ([\Lambda],[R]) R_0^{-\frac{1}{2}n}$$
 (D.2)

as a series in R_1 and R_2 {noting that R_i is of the order ℓ_0^i/r_{110}^2 . Since R_0 is 1, the last factor contributes nothing to the expression. However, its introduction makes it easier to generalize the equations being developed here to wire-to-surface and surface-to-surface equations.

The application of the binomial theorem gives

$$\left(R_{0} + R_{1}x + R_{2}x^{2}\right)^{\frac{1}{2}Q} = R_{0}^{\frac{1}{2}Q} \left(1 + \frac{x}{R_{0}} \left(R_{1} + R_{2}x\right)\right)^{\frac{1}{2}Q}$$

$$= R_{0}^{\frac{1}{2}Q} \sum_{\beta=0}^{\infty} \left(\frac{1}{\beta!} \prod_{\delta=1}^{\beta} \left(\frac{1}{2}Q - (\delta-1)\right)\right) \left(\frac{x}{R_{0}}\right)^{\beta} \left(R_{1} + R_{2}x\right)^{\beta}$$

$$= \sum_{\beta=0}^{\infty} \left(\frac{1}{\beta!} \prod_{\delta=1}^{\beta} \left(\frac{1}{2}Q - (\delta-1)\right)\right) R_{0}^{\frac{1}{2}Q - \beta} \sum_{\epsilon=0}^{\beta} \left(\frac{\beta}{\epsilon}\right) R_{1}^{\beta-\epsilon} R_{2}^{\epsilon} x^{\beta+\epsilon}$$

$$(D.3)$$

The results of equation D.3 are used for rewriting the equations for $2_0^{\bullet\bullet}$ in a form that is numerically rugged when $\ell<<$ r. Thus the quantity under consideration in equation D.2 becomes

$$\int_{0}^{1} \left(\Lambda(\mathbf{x})\right)^{\frac{1}{2}\nu} \mathbf{x}^{\mathbf{s+m'}} \int_{0}^{\infty} \sum_{\epsilon_{0}=0}^{\infty} \left[\mathbf{q} \left[\frac{1}{\epsilon_{0}^{1}} \prod_{\delta_{0}=1}^{\beta_{0}} \left(\frac{1}{\epsilon_{0}^{1}} \prod_{\delta_{0}=1}^{\beta_{0}} \prod_{\delta_{0}=1}^{\beta_{0}} \left(\frac{1}{\epsilon_{0}^{1}} \prod_{\delta_{0}=1}^{\beta_{0}} \prod_{\delta_{0}=1}^{\beta_{0}} \prod_{\delta_{0}=1}^{\beta_{0}} \prod_{\delta_{0}=1}^{\beta_{0}} \left(\frac{1}{\epsilon_{0}^{1}} \prod_{\delta_{0}=1}^{\beta_{0}} \prod_{$$

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The quantity in the square brackets is empirically found to be zero* for $\beta_0 < q$. Thus the expression D.4 can be written, (using $\beta_0' = \beta_0 - q$),

$$\times A_{\alpha q \beta_0} Z_{A2 \vee (s+m + \beta_0^{\dagger} + \epsilon_0 + q) 0} ([\Lambda], [R])$$
(D.5)

*A formal proof was not attempted in this effort. However, that the summation in expression D.2 should be of the order $\left(\frac{\ell_0}{r_{110}}\right)^q$ becomes obvious if one writes

$$\frac{q}{\sum_{n=0}^{\infty} {n \choose n} (-1)^n} Z_{A2\nu (s+m^i) (n+\alpha)}$$

$$= \int_{0}^{1} (\Lambda(x))^{\frac{1}{2}\nu} x^{s+m^i} (R(x))^{\frac{1}{2}\mu} \begin{bmatrix} q & q \\ 0 & n=0 \end{bmatrix} (-1)^n (R^{\frac{1}{2}}(x))^n dx.$$

All factors except the square brackets are of order 1. The quantity inside the square parenthesis is, by applying the binomial theorem in reverse,

$$(1 - R^{\frac{1}{2}}(x))^{q} = \left[1 - \left[1 + \frac{1}{2} \left(R_{1} + R_{2} x\right)x\right]\right]^{q} = \left[-\frac{1}{2} R_{1}x - \frac{1}{2} R_{2}x^{2}\right]^{q}$$

$$= \left[\frac{\ell_{0}}{r_{110}}\right]^{q} \left[-\frac{1}{2} \left(\frac{R_{1} r_{110}}{\ell_{0}}\right)x - \frac{1}{2} \left(\frac{R_{2} r_{110}}{\ell_{0}}\right)x^{2}\right]^{q}$$

The quantity in the second factor is of the order 1. Thus the quantity in square brackets is of the order $\left(\frac{\ell_\sigma}{r_{110}}\right)^q$.

where

$$A_{\alpha q \beta 0} = \sum_{n=0}^{q} {q \choose n} (-1)^n \frac{1}{(\beta 0 + q)!} \frac{\beta 0 + q}{\delta 0 = 1} \begin{bmatrix} \frac{1}{2} (n + \alpha) - (\delta_0 - 1) \end{bmatrix} \text{ if } \beta 0 > 0$$

$$= 0 \qquad \qquad \text{Otherwise (D.6)}$$

The quantity inside the summation in expression D.5 is of the order $\left(\frac{\ell_{\sigma}}{r_{110}}\right)^{\beta_0'+\epsilon_0}$. Writing expression D.6 indexed on $\eta_0 = \beta_0' + \epsilon_0$ and truncating the infinite series after $\eta_0 = 2$

Thus for $\ell_{\sigma} \ll r_{11}$, equation D.1 is implemented using

$$z_{0}^{WW} = \sum_{\sigma=1}^{2} \frac{(-1)^{\sigma s}}{2j \sin k \ell_{\sigma}} \frac{\exp(-jkr_{110}) \ell_{\sigma}^{s+1} (\Lambda_{0}^{\varsigma})^{\frac{1}{2}\nu} r_{110}^{\alpha}}{2j \sin k \ell_{\sigma}}$$

$$x \sum_{i=0}^{5} a_{i} \left(jk \ell_{\sigma}\right)^{i} \sum_{q=0}^{i} \left(1 - (-1)^{i+q}\right) \sum_{m'=0}^{i-q} \left(i \atop m'+q\right) \left(m'+q \atop m'\right) (-1)^{m'}$$

$$x \left[\frac{R_1 r_{110}}{\ell_{\sigma}} \right]^{q} \sum_{\substack{n_0 = 0 \\ n_0 = 0}}^{2} \left(\frac{\ell_{\sigma}}{r_{110}} \right)^{n_0} \sum_{\substack{\epsilon_0 = 0 \\ \epsilon_0 = 0}}^{\frac{1}{2} \left(n_0 + q \right)} \left(\frac{q + n_0 - \epsilon_0}{\epsilon_0} \right) \left(\frac{R_1 r_{110}}{\ell_{\sigma}} \right)^{n_0 - 2\epsilon_0}$$

$${}^{\mathbf{X}} {}^{\mathbf{A}} {}_{\alpha \mathbf{q}} (\eta_0 - \varepsilon_0) {}^{\mathbf{Z}} {}^{\mathbf{A}} {}^{\mathbf{2}} \mathbf{v} (\mathbf{s} + \mathbf{m}^{\mathbf{t}} + \eta_0 + \mathbf{q}) 0$$
 (D.8)

. .

The above equation has been written so as to keep all partial sums independent of the ratio $\frac{\ell_\sigma}{r_{110}}$ which may vary considerably in the computations depending upon the geometry. Thus all partial sums in the above equation are well behaved.

Computation of wire-to-surface and surface-to-wire impedances is performed using a strategy similar to one used for computation of wire-to-wire impedances. Equation 2.4.17a is written as

$$z_{0e}^{WS} = \sum_{\sigma=1}^{2} \frac{(-1)^{\sigma s} \exp(-jkr_{010}) \ell_{\sigma}^{s+1} \left(\Lambda_{00}^{\zeta Y'} \right)^{\frac{k_{2}\nu}{2\nu}} r_{010}^{\alpha}}{2j \sin k \ell_{\sigma}}$$

$$\times \sum_{q=0}^{i} \left[1 - (-1)^{i+q}\right] \left(\frac{r_{010}}{\ell_{\sigma}}\right)^{q} \sum_{m'=0}^{i-q} \left(\frac{i}{m'+q}\right) \left(\frac{m'+q}{m'}\right) (-1)^{m'}$$

$$\begin{array}{ccc}
\mathbf{x} & \overset{\mathbf{q}}{\overset{\Sigma}{\Sigma}} & \begin{pmatrix} \mathbf{q} \\ \mathbf{n} \end{pmatrix} & (-1)^{n} & \overset{\mathbf{i}_{1}}{\overset{\Sigma}{\Sigma}} & \begin{pmatrix} \mathbf{i}_{1} \\ \mathbf{m}_{1} \end{pmatrix} (-1)^{m_{1}} \\
& & & & & & & & & \\
\end{array}$$

$$\times Z_{\underline{B}2\nu(s+m')(n+\alpha)e(q-n+m_1)}([\Lambda],[R])$$
 (D.9)

It is noted that the variable n occurs in two of the Z_{B2},\ldots indices. This makes the application of recursion relations B.2.2, B.2.3 and B.2.4 inconvenient as the summation is executed.* The summation is made more convenient by writing equation D.9 to give

$$\begin{split} z_{0e}^{WS} &= \sum_{\sigma=1}^{2} \frac{(-1)^{\sigma s} \exp(-jkr_{010}) \ell_{\sigma}^{s+1} (\Lambda_{00}^{\zeta y'})^{\frac{1}{2}\nu} r_{010}^{\alpha}}{2j \sin k\ell_{\sigma}} \\ x &= \sum_{i=0}^{5} a_{i} (jk\ell_{\sigma})^{i} \sum_{n=0}^{i} (-1)^{n} \sum_{q=n}^{i} (1-(-1)^{i+q}) (\frac{r_{010}}{\ell_{\sigma}})^{q} r_{q}^{q} \\ x &= \sum_{m'=0}^{5} \left[\prod_{m'+q}^{i} \left[\prod_{m'}^{a'+q} (-1)^{m'} \sum_{m_{1}=0}^{5} (-1)^{m_{1}} \sum_{i_{1}=m_{1}}^{5} a_{i_{1}} (jk\ell_{\tau}^{i})^{i_{1}} \right] \\ x &= (\frac{r_{010}}{\ell_{\tau}^{i}})^{i_{1}} \left[\prod_{m_{1}}^{i_{1}} \right] z_{B2\nu(s+m')(n+\alpha)e} (q-n+m_{1}^{i}) \end{split}$$

^{*}There are two approaches to evaluating the expression D.9. The first is to fill a 5 dimensional matrix with the appropriate values of $\mathbf{Z}_{\mathrm{B2},\ldots}$ and compute according to equation D.9. This approach is attractive but would take approximately 2K of core for storing $\mathbf{Z}_{\mathrm{B2},\ldots}$ and approximately 100K for storing $\mathbf{Z}_{\mathrm{C2},\ldots}$ in computing $\mathbf{Z}_{0}^{\mathrm{SS}}$. Since this is too large, a second approach, that of calculating the base values of $\mathbf{Z}_{\mathrm{B2},\ldots}$ (and storing them) and then using equations B.2.2, B.2.3 and B.2.4 as the summation proceeds, must be used.

or, writing $q-n+m_1=n_1$ as a summation index,

$$z_{0e}^{WS} = \sum_{\sigma=1}^{2} \frac{(-1)^{\sigma s} \exp(-jkr_{010}) \ell_{\sigma}^{s+1} (\Lambda_{00}^{\zeta y'})^{\frac{1}{2}v} r_{010}^{\alpha}}{2j \sin k\ell_{\sigma}}$$

$$\times (1-(-1)^{i+n+n}1^{-m}1) \binom{n+n}{n}1^{-m}1 (-1)^{m}1 \binom{r_{010}}{\ell_{\sigma}}^{n+n}1^{-m}1$$

$$x \int_{1_{1}=m_{1}}^{5} \left(\frac{i}{m_{1}} \right) a_{i_{1}} (jk \ell_{\tau}')^{i_{1}} \left(\frac{r_{010}}{\ell_{\tau}^{1}} \right)^{i_{1}} Z_{B2\nu(s+m')(n+\alpha)en_{1}}$$
(D.10)

As in the wire-to-wire case, it is appropriate and convenient to express the quantity

(D.11)

in equation D.9 as a series in $(\frac{\ell_{\sigma}}{r_{010}})$ when $\ell_{\sigma} << r_{010}$.

In the wire-to-surface case, R_{00} is 1.

The last factor has been included to facilitate generalization of this development to the surface-to-surface case. Expression D.11 can be written

$$\frac{i_{1}}{\sum_{m_{1}=0}^{E} {i_{1} \choose m_{1}}} (-1)^{m_{1}} R_{00}^{-\frac{1}{2}m_{1}} \int_{0}^{1} y^{e} \left(R_{0}(y)\right)^{\frac{1}{2}(q+m_{1})} \\
\times \left[\sum_{n=0}^{q} {q \choose n} (-1)^{n} \left(R_{0}(y)\right)^{-\frac{1}{2}n} \int_{0}^{1} \left(\Lambda(x,y)\right)^{\frac{1}{2}y} x^{s+m'} \left(R(x,y)\right)^{\frac{1}{2}(n+\alpha)} dx \right] dy$$

The term inside the square brackets is recognized as expression D.2 and is replaced by its equal expression D.7 to give

$$\frac{i_{1}}{\Sigma} \left(\frac{i_{1}}{m_{1}} \right) (-1)^{m_{1}} \frac{-i_{2}m_{1}}{R_{00}} \frac{2}{n_{0}=0} \frac{i_{2}(n_{0}+q)}{\Sigma} \left(\frac{q+n_{0}-\varepsilon_{0}}{\varepsilon_{0}} \right) A_{\alpha q(n_{0}-\varepsilon_{0})}$$

$$\times \int_{0}^{1} y^{e} \left(R_{0}(y) \right)^{\frac{i_{2}}{2}(q+m_{1}+\alpha)-(q+n_{0}-\varepsilon_{0})} \left(R_{1}(y) \right)^{q+n_{0}-2\varepsilon_{0}} R_{20}^{\varepsilon_{0}}$$

$$\times \int_{0}^{1} \left(\Lambda(x,y) \right)^{\frac{i_{2}}{2}\nu} x^{s+m'+n_{0}+q} dx dy \qquad (D.12)$$

. .

or by a development process similar to that used in developing (D.7) from (D.2),

$$\begin{array}{ccc} 2 & \frac{1}{2}(\eta_0+q) \\ \Sigma & \Sigma \\ \eta_0=0 & \varepsilon_0=0 \end{array} & \begin{pmatrix} q+\eta_0-\varepsilon_0 \\ \varepsilon_0 \end{pmatrix} & \begin{array}{cccc} A_{\alpha q}(\eta_0-\varepsilon_0) & \frac{\varepsilon}{R} 0 & R_1^{q+\eta_0-2\varepsilon_0} \end{array}$$

$$\times {}^{\mathbf{Z}}B2v(s+m'+q+\eta_0)0(e+\gamma_0+i_1+\eta_1)0$$
 (D.14)

. 1

Thus, for ℓ_{σ} , ℓ_{τ}^{\prime} << r_{010} , equation D.9 becomes

$$z_{0e}^{WS} = \frac{1}{2j} \exp(-jkr_{010}) \left(\Lambda_{00}^{\zeta Y'}\right)^{\frac{1}{2}\nu} \qquad \qquad r_{010}^{\alpha} \quad \sum_{\sigma=1}^{2} \frac{(-1)^{\sigma s}}{\sin k \ell_{\sigma}} \quad \ell_{\sigma}^{s+1}$$

$$\begin{array}{ccc}
 & i-q & i \\
 & \Sigma & m'=0 & m'+q & m' \\
\end{array}$$

$$x_{\gamma_{0}=0}^{q+\eta_{0}-2\varepsilon_{0}} \left(\begin{array}{c} q^{+\eta_{0}-2\varepsilon} \\ \gamma_{0} \end{array} \right) \left(\frac{\imath_{\tau}}{r_{010}} \right)^{\gamma_{0}} \left(\frac{R_{10} r_{010}}{\imath_{\sigma}} \right)^{q+\eta_{0}-2\varepsilon_{0}-\gamma_{0}} \left(\frac{R_{11} r_{010}^{2}}{\imath_{\sigma} \imath_{\tau}} \right)^{\gamma_{0}} \left(\frac{R_{11} r_{010}^{2}}{\imath_{\sigma}} \right)^{\gamma_{0}} \left(\frac{R_{1$$

$${}^{\times} \quad {}^{A}(\alpha+q-2(q+\eta_{0}-\epsilon_{0}))i_{1}(\eta_{1}-\epsilon_{1}) \quad {}^{Z}B2\nu(s+m'+q+\eta_{0})0(e+\gamma_{0}^{+}i_{1}+\eta_{1})0$$

(D.15a)

It is to be noted that all partial sums in equation D.15a are well behaved. Similarly,

$$\begin{split} z_{0f}^{SW} &= \frac{1}{2j} \, \exp \left(-jk r_{100} \right) \left(\Lambda_{00}^{\zeta \eta} \right)^{\frac{1}{2}\nu} & r_{100}^{\alpha} \, \frac{2}{\sigma = 1} \, \frac{\left(-1 \right)^{\sigma S}}{\sin k \, \ell_{\sigma}} \, \ell_{\sigma}^{S+1} \\ &\times \, \sum\limits_{i=0}^{5} \, a_{i} \, \left(jk \ell_{\sigma} \right)^{i} \, \sum\limits_{i_{1}=0}^{5} \, a_{i_{1}} \left(jk \ell_{\xi} \right)^{i_{1}} \, \sum\limits_{q=0}^{i} \, \left(1 - \left(-1 \right)^{i+q} \right) \\ &\times \, \sum\limits_{m'=0}^{i-q} \, \left(\frac{i}{m' + q} \right) \left(\frac{m' + q}{m'} \right) \left(-1 \right)^{m'} \\ &\times \, \sum\limits_{m'=0}^{2} \, \left(\frac{\ell_{\sigma}}{r_{101}} \right)^{n_{0}} \, \sum\limits_{\epsilon_{0}=0}^{i_{2} (n_{0} + q)} \, \left(\frac{q + n_{0} - \epsilon_{0}}{\epsilon_{0}} \right) \, A_{\alpha q \, (n_{0} - \epsilon_{0})} \\ &\times \, \sum\limits_{\gamma_{0}=0}^{q + \gamma_{0} - 2\epsilon_{0}} \, \left(\frac{q + \eta_{0} - 2\epsilon_{0}}{\gamma_{0}} \right) \, \left(\frac{\ell_{\xi}}{r_{100}} \right)^{\gamma_{0}} \, \left(\frac{R_{10} \, r_{100}}{\ell_{\sigma}} \right) \, q^{q + \eta_{0} - 2\epsilon_{0} - \frac{\gamma}{2}_{0}} \, \left(\frac{R_{11} \, r_{100}^{2}}{\ell_{\sigma} \, \ell_{\xi}} \right)^{\gamma_{0}} \\ &\times \, \sum\limits_{\gamma_{1}=0}^{2} \, \left(\frac{\ell_{\xi}}{r_{100}} \right)^{n_{1}} \, \frac{\ell_{2} (n_{1} + i_{1})}{\epsilon_{1}^{2}} \, \left(\frac{i_{1} + \eta_{1} - \epsilon_{1}}{\epsilon_{1}} \right) \, \left(\frac{R_{01} \, r_{100}}{\ell_{\xi}} \right)^{i_{1} + \eta_{1} - 2\epsilon_{1}} \\ &\times \, \sum\limits_{\gamma_{1}=0}^{2} \, \left(\frac{\ell_{\xi}}{r_{100}} \right)^{n_{1}} \, \frac{\ell_{2} (n_{1} + i_{1})}{\epsilon_{1}^{2}} \, \left(\frac{i_{1} + \eta_{1} - \epsilon_{1}}{\epsilon_{1}} \right) \, \left(\frac{R_{01} \, r_{100}}{\ell_{\xi}} \right)^{i_{1} + \eta_{1} - 2\epsilon_{1}} \end{split}$$

 X A $(\alpha-2\eta_0+2\varepsilon_0-q)i_1(\eta_1-\varepsilon_1)$ Z B2 $\nu(s+m'+q+\eta_0)0(f+\gamma_0+i_1+\eta_1)0$

a street and

(D.15b)

For the surface-to-surface case, combining equations 2.5.6 and 2.5.9,

$$z_{\text{0ef}}^{\text{SS}} = \sum_{\sigma=1}^{2} \frac{(-1)^{\sigma \text{S}} \exp(-jkr_{000}) \ell_{\sigma}^{\text{S+1}} (\Lambda_{000}^{\zeta y' \eta})^{\frac{1}{2} \nu} r_{000}^{\alpha}}{2j \sin k \ell_{\sigma}}$$

$$x \sum_{i=0}^{5} a_{i}(jk\ell_{\sigma})^{i} \sum_{i_{1}=0}^{5} a_{i_{1}}(jk\ell_{\tau}^{c})^{i_{1}} \left(\frac{r_{000}}{\ell_{\tau}^{c}}\right)^{i_{1}} \sum_{i_{2}=0}^{5} a_{i_{2}}(jk\ell_{\xi})^{i_{2}} \left(\frac{r_{000}}{\ell_{\xi}}\right)^{i_{2}}$$

$$\times \sum_{q=0}^{i} \left\{ 1 - (-1)^{i+q} \right\} \left(\frac{r_{000}}{\ell_{\sigma}} \right)^{q} \quad \sum_{m'=0}^{i-q} \quad \left\{ i \atop m'+q \right\} \quad \left\{ m'+q \atop m' \right\} \left(-1\right)^{m'}$$

$$\times Z_{C2v(s+m')(n+\alpha)e(q-n+m_1)f(i_1-m_1+m_2)}$$
 ([A],[R]) (D.16)

or, similar to equation D.10,

$$\begin{split} z_{0ef}^{SS} &= \frac{2}{\sigma = 1} \frac{(-1)^{\sigma S} \exp(-jkr_{000}) \frac{\ell_{\sigma}^{S+1} (\Lambda_{000}^{\zeta y' \eta})^{\frac{1}{2} \nu} r_{000}^{\alpha}}{2j \sin k\ell_{\sigma}} \\ \times \frac{5}{i=0} a_{i} (jk\ell_{\sigma})^{i} \frac{i}{\sum_{n=0}^{\infty} (-1)^{n} \frac{5+i-n}{\sum_{n=0}^{\infty} m_{1}=max(0,n_{1}+n-i)}}{n_{1}=0 m_{1}=max(0,n_{1}+n-i)} \\ \times \left(1-(-1)^{i+n+n}1^{-m}1\right) \binom{n+n}{n}1^{-m}1 \left(-1\right)^{m_{1}} \left(\frac{r_{000}}{\ell_{\sigma}}\right)^{n+n_{1}-m}1 \\ \times \frac{i-(n+n_{1}-m_{1})}{\sum_{m'=0}^{\infty} m'=0} \binom{i}{m'+n+n_{1}-m_{1}} \binom{m'+n+n_{1}-m}{m'} \binom{n-1}{m'} \frac{m'}{m'} \\ \times \frac{5}{i} \binom{i}{m} a_{i} (jk\ell_{\sigma}^{i})^{i} \binom{r_{000}}{\ell_{\sigma}^{i}} \frac{1}{m_{2}=0} \binom{-1}{m}^{m_{2}} \frac{5}{i} \binom{i}{2m_{2}} \\ \times \frac{5}{i} \binom{i}{m} a_{i} \binom{n}{i} \binom{n}{i} \binom{n}{\ell_{\sigma}^{i}} \frac{1}{\ell_{\sigma}^{i}} \binom{n}{\ell_{\sigma}^{i}} \frac{1}{m_{2}} \binom{n}{\ell_{\sigma}^{i}} \frac{1}{m_{2}} \binom{n}{\ell_{\sigma}^{i}} \frac{1}{m_{2}^{i}} \binom{n}{\ell_{\sigma}^{i}} \frac{1}{m_{2}^{i}} \binom{n}{\ell_{\sigma}^{i}} \frac{1}{m_{2}^{i}} \binom{n}{\ell_{\sigma}^{i}} \frac{1}{m_{2}^{i}} \binom{n}{\ell_{\sigma}^{i}} \frac{1}{m_{2}^{i}} \binom{n}{\ell_{\sigma}^{i}} \binom{n}{\ell_{\sigma}^{i$$

$$Z_{0ef}^{SS} = \sum_{\sigma=1}^{2} \frac{(-1)^{\sigma s} \exp(-jkr_{000}) \ell_{\sigma}^{s+1} (\Lambda_{000}^{\zeta y' \eta})^{\frac{1}{2}\nu} r_{000}^{\alpha}}{2j \sin k \ell_{\sigma}}$$

$$\times \sum_{i=0}^{5} a_{i}^{(jk\ell_{\sigma})} \sum_{n=0}^{i} (-1)^{n} \sum_{n=0}^{5+i-n} \min_{n_{1}=0}^{min} (5, n_{1}) \sum_{m=0}^{5} m_{1}^{m} \sum_{n=0}^{5+i-n} m_{1}^{m} \sum_{m=0}^{5} m_{1}^{m} \sum_{m=0}^{5+i-n} m_{1}^{m} \sum_{m=0}^{$$

When r_{000} is large, the innermost summations in (D.16) are evaluated by considering the expression

$$x^{Z}_{C2\nu}(s+m')(n+\alpha)e(q-n+m_1)f(i_1-m_1+m_2)$$

$$= \sum_{m_{2}=0}^{i_{2}} {i_{2} \choose m_{2}} (-1)^{m_{2}} \int_{0}^{1} z^{f} \left(R_{00}(z)\right)^{\frac{1}{2}(i_{1}+m_{2})} \begin{bmatrix} i_{1} \\ \Sigma \\ m_{1}=0 \end{bmatrix} {i_{1} \choose m_{1}} (-1)^{m_{1}} \left(R_{00}(z)\right)^{\frac{1}{2}m_{1}} \times \int_{0}^{1} y^{e} \int_{0}^{\infty} \left(R_{00}(z)\right)^{m_{1}} \left(R_$$

$$x \int_{0}^{1} \left(\Lambda(x,y,z) \right)^{\frac{1}{2}\nu} x^{s+m'} \left(R(x,y,z) \right)^{\frac{1}{2}(n+\alpha)} dx dy dz$$

$$\left(D.18 \right)$$

. .

The term inside the square brackets is recognized as expression D.11 and is replaced by its equal expression D.14 to give

$$(R_{10}(z))^{q+\eta_0-2\epsilon_0} \times \sum_{\gamma_0} {q+\eta_0-2\epsilon_0 \choose \gamma_0} (R_{10}(z))^{-\gamma_0} (R_{11}(z))^{\gamma_0} R_{01}$$

$$\begin{array}{ccc} \mathbf{x} & \sum & \sum & \left\{ \mathbf{i}_1 + \mathbf{n}_1 - \varepsilon_1 \\ & \mathbf{n}_1 & \varepsilon_1 \end{array} \right\} & \mathbf{A} \left(\mathbf{q} + \alpha - 2 \left(\mathbf{q} + \mathbf{n}_0 - \varepsilon_0 \right) \right) \mathbf{i}_1 \left(\mathbf{n}_1 - \varepsilon_1 \right) \end{array}$$

$$\times (R_{00}(z)^{\frac{1}{2}(q+\alpha)-(q+\eta_{0}-\varepsilon_{0})-(i_{1}+\eta_{1}-\varepsilon_{1})} R_{01}(z))^{\eta_{1}-2\varepsilon_{1}} (R_{02}(z))^{\varepsilon_{1}}$$

Where the limits of the summations are the same as have been noted before. The above expression is rewritten as

$$\begin{array}{c} \sum\limits_{\eta_{0}}\sum\limits_{\epsilon_{0}}\left(\frac{q+\eta_{0}-\epsilon_{0}}{\epsilon_{0}}\right) & A_{\alpha q}(\eta_{0}-\epsilon_{0}) & R_{2}^{\epsilon}g_{0} \\ \\ \sum\limits_{\gamma_{0}}\left(\frac{q+\eta_{0}-2\epsilon_{0}}{\gamma_{0}}\right) & \frac{q+\eta_{0}-2\epsilon_{0}-\gamma_{0}}{\gamma_{1}=0} & \frac{q+\eta_{0}-2\epsilon_{0}-\gamma_{0}}{\gamma_{1}} & \frac{q+\eta_{0}-2\epsilon_{0}-\gamma_{0}-\gamma_{1}}{\gamma_{1}} & \gamma_{0} \\ \\ \sum\limits_{\eta_{1}}\sum\limits_{\epsilon_{1}}\left(\frac{i_{1}+\eta_{1}-\epsilon_{1}}{\epsilon_{1}}\right) & A_{(q+\alpha-2(q+\eta_{0}-\epsilon_{0}))i_{1}(\eta_{1}-\epsilon_{1})} \\ \\ \sum\limits_{\eta_{1}}\sum\limits_{\epsilon_{1}}\left(\frac{i_{1}+\eta_{1}-2\epsilon_{1}}{\gamma_{2}}\right) & A_{(q+\alpha-2(q+\eta_{0}-\epsilon_{0}))i_{1}(\eta_{1}-\epsilon_{1})} \\ \\ \sum\limits_{\gamma_{2}=0}\sum\limits_{\gamma_{2}}\left(\frac{i_{1}+\eta_{1}-2\epsilon_{1}}{\gamma_{2}}\right) & R_{010} & R_{01}^{\gamma_{1}} & R_{020}^{\epsilon_{1}} \\ \\ \times \sum\limits_{\eta_{2}=0}\sum\limits_{\gamma_{2}}\left(\frac{i_{2}}{\eta_{2}}\right) & A_{010} & R_{01}^{\gamma_{2}}(\alpha+q+i_{1})-(q+\eta_{0}-\epsilon_{0})-(i_{1}+\eta_{1}-\epsilon_{1}) \\ \\ \times \sum\limits_{\eta_{2}=0}\sum\limits_{\gamma_{2}}\left(\frac{i_{2}}{\eta_{2}}\right) & A_{010} & R_{01}^{\gamma_{2}}(\alpha+q+i_{1})-(q+\eta_{0}-\epsilon_{0})-(i_{1}+\eta_{1}-\epsilon_{1}) \\ \\ \times \sum\limits_{\eta_{2}=0}\sum\limits_{\gamma_{2}}\left(\frac{i_{2}}{\eta_{2}}\right) & A_{010} & R_{010}^{\gamma_{2}}(\alpha+q+i_{1})-(q+\eta_{0}-\epsilon_{0})-(i_{1}+\eta_{1}-\epsilon_{1}) \\ \\ \times \sum\limits_{\eta_{2}=0}\sum\limits_{\gamma_{2}}\left(\frac{i_{2}}{\eta_{2}}\right) & A_{010} & R_{010}^{\gamma_{2}}(\alpha+q+i_{1})-(q+\eta_{0}-\epsilon_{0})-(i_{1}+\eta_{1}-\epsilon_{1}) \\ \\ \times \sum\limits_{\eta_{2}=0}\sum\limits_{\gamma_{2}}\left(\frac{i_{2}}{\eta_{2}}\right) & A_{010} & R_{010}^{\gamma_{2}}(\alpha+q+i_{1})-(q+\eta_{0}-\epsilon_{0})-(i_{1}+\eta_{1}-\epsilon_{1}) \\ \\ \times \sum\limits_{\eta_{2}=0}\sum\limits_{\gamma_{2}}\left(\frac{i_{2}}{\eta_{2}}\right) & A_{010} & A_{010}^{\gamma_{2}}(\alpha+q+i_{1})-(q+\eta_{0}-\epsilon_{0})-(i_{1}+\eta_{1}-\epsilon_{1}) \\ \\ \times \sum\limits_{\eta_{2}=0}\sum\limits_{\gamma_{2}}\left(\frac{i_{2}}{\eta_{2}}\right) & A_{010}^{\gamma_{2}}(\alpha+q+i_{1})-(q+\eta_{0}-\epsilon_{0})-(i_{1}+\eta_{1}-\epsilon_{1}) \\ \\ \times \sum\limits_{\eta_{2}=0}\left(\frac{i_{2}}{\eta_{2}}\right) & A_{010}^{\gamma_{2}}(\alpha+q+i_{1})-(q+\eta_{0}-\epsilon_{0}) \\ \\ \times \sum\limits_{\eta_{2}=0}\left(\frac{i_{2}}{\eta_{2}$$

This expression is developed similar to the development of expression D.7 to give

. .

$$\begin{array}{ccc} \Sigma & \Sigma & \begin{pmatrix} q + \eta_0 - \epsilon_0 \\ \epsilon_0 \end{pmatrix} & A_{\alpha q (\eta_0 - \epsilon_0)} & R_{200}^{\epsilon} \end{array}$$

$$\begin{array}{c|c} \Sigma & \left\{\begin{matrix} q+n_0-2\epsilon_0 \\ \gamma_0 \end{matrix}\right\} & \begin{array}{c} q+n_0-2\epsilon_0-\gamma_0 \\ \Sigma \\ \gamma_1=0 \end{array} & \left\{\begin{matrix} q+n_0-2\epsilon_0-\gamma_0 \\ \gamma_1 \end{matrix}\right\} \begin{matrix} q+n_0-2\epsilon_0-\gamma_0-\gamma_1 \\ R_{100} \end{matrix} & \begin{array}{c} \gamma_1 \\ R_{110} \end{matrix} \\ \end{array}$$

$$\begin{array}{ccc} \Sigma & \Sigma & \left[i_1+\eta_1-\varepsilon_1\right] & A & \left(q+\alpha-2\left(q+\eta_0-\varepsilon_0\right)\right)i_1\left(\eta_1-\varepsilon_1\right) \\ \varepsilon_1 & \end{array}$$

$$\begin{array}{cccc} 2 & \frac{1}{2}(\eta_2 + i_2) & \begin{bmatrix} i_2 + \eta_2 - \epsilon_2 \\ \epsilon_2 \end{bmatrix} & \frac{i_2 + \eta_2 - 2\epsilon_2}{R_{001}} & \frac{\epsilon_2}{R_{002}} \end{array}$$

$$\overset{\mathbf{x}}{\overset{\mathbf{A}}{(\mathtt{i}_1+\mathtt{q}+\alpha-2(\mathtt{q}+\mathtt{n}_0-\epsilon_0)-2(\mathtt{i}_1+\mathtt{n}_1-\epsilon_1))}\mathtt{i}_2(\mathtt{n}_2-\epsilon_2)}$$

Thus, for the far-field case, equation D.16 is written as

$$z_{0ef}^{SS} = \sum_{\sigma=1}^{2} \frac{(-1)^{\sigma s} \exp(-jkr_{000}) \ell_{\sigma}^{s+1} \left(\Lambda_{000}^{\zeta y' \eta} \right)^{\frac{1}{2} \nu} r_{000}^{\alpha}}{2j \sin k \ell_{\sigma}}$$

$$\stackrel{i}{\underset{q=0}{\Sigma}} \left(1 - (-1)^{i+q}\right) \stackrel{i-q}{\underset{m'=0}{\Sigma}} \left(\stackrel{i}{\underset{m'+q}{\prod}} \left(\stackrel{i}{\underset{m'}{\prod}} + \stackrel{i}{\underset{m'}{\prod}} \right) (-1)^{m'} \quad \left(\frac{R_{100}r_{000}}{\ell_{\sigma}} \right)^{q} \quad \left(\frac{R_{010}r_{000}}{\ell_{\tau}} \right)^{i_{1}} \quad \left(\frac{R_{001}r_{000}}{\ell_{\xi}} \right)^{i_{2}}$$

$$\sum_{\substack{n_0=0\\ n_0=0}}^{2} \left(\frac{\iota_{\sigma}}{r_{000}}\right)^{n_0} \sum_{\epsilon_0=0}^{\frac{\iota_2(q+n_0)}{\Sigma}} \left(\frac{q+n_0-\epsilon_0}{\epsilon_0}\right) A_{\alpha q(n_0-\epsilon_0)} \left(\frac{R_{100} r_{000}}{\iota_{\sigma}}\right)^{n_0-2\epsilon_0}$$

$$x = \left[\frac{R_{101}r^2000}{r_0 r_{\xi}} \right]^{\gamma} \left[\frac{R_{110}r^2000}{r_0 r_{\tau}^2} \right]^{\gamma} 0$$

(Continued...)

$$\times \ ^{\mathrm{A}}(\mathtt{i}_{1}+\mathtt{q}+\alpha-2(\mathtt{q}+\mathtt{n}_{0}-\epsilon_{0})-2(\mathtt{i}_{1}+\mathtt{n}_{1}-\epsilon_{1}))\,\mathtt{i}_{2}(\mathtt{n}_{2}-\epsilon_{2})$$

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